

# Pseudotensor Problem of Gravitational Energy-momentum and Noether's Theorem Revisited

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## Abstract

Based on a general variational principle, Noether's theorem is revisited. It is shown that the so called pseudotensor problem of the gravitational energy-momentum is a result of mis-reading Noether's theorem, and in fact, all the Noether's conserved quantities are scalars. As a side product, a generalized Hamilton-Jacobi equation for the Hamilton's principal functional is obtained.

## 1 Introduction

Ever since Einstein introduced the gravitational energy-momentum complex to keep the law of conservation of energy-momentum alive in general relativity (GR) in 1918, the so-called pseudotensor problem of various gravitational energy-momentum complexes has been troubling relativitists [1]. It is widely accepted that the proof of the positivity of the total gravitational energy both in spacial and null infinity is one of the greatest achievements in classical GR in the last 30 years. This success inspired efforts to find quasi-local gravitational energy-momentum. Non-locality is definitely not acceptable; however, finding quasi-local gravitational quantities has proven to be surprisingly difficult [2]. Experimentally, the gravitational wave carrying energy-momentum has not been tested[3]. The aim of this article is to explore the pseudotensor

problem of the gravitational energy-momentum, in particular, the difficulty of non-locality of the gravitational energy-momentum. The whole argument is based on the variational principle of dynamics. It is pointed out that the so-called pseudotensor problem of the gravitational energy-momentum is only a result of mis-reading Noether's theorem, and mistaking different geometric physical objects as one and the same; hence the non-locality difficulty does not exist at all. In fact, all Noether's conserved quantities are scalars. As a side product, a generalized Hamilton-Jacobi equation for the Hamilton's principal functional is obtained.

The history of modern physics has proven that the variational principle approach to dynamics is not only an alternative and equivalent version to the naive, intuitive approach, but also yields deeper insights into the underlying physics. For instance, it is hard to imagine that the statistical mechanics could have been established without using the concept of phase space, and that the quantum mechanics could have been established without using the concept of Hamiltonian. Therefore, we will base our argument on a general variational principle for the classical field. It might be for the similar consideration that soon after Einstein proposed his general theory of relativity, Hilbert made the first successful attempt to get Einstein's equation by using the least action principle. The Lagrangian being used for vacuum Einstein's equation,  $(16\pi G)^{-1}R$ , is the only independent scalar constructed in terms of the metric field and its derivatives of no higher than the second order. However, because the Ricci scalar curvature  $R$  contains the second order derivatives of the metric field  $g_{\mu\nu}(x)$ , which is now the dynamic variable along with the matter field, the least action principle for Lagrangians containing only the field quantity and its first order derivatives does not lead to Einstein's field equation. The generally accepted solution to this difficulty is adding the Gibbons-Hawking boundary term to the Hilbert action and keeping the least action principle unchanged[4]. But there is another solution to this difficulty, which was initiated by Hilbert and will be adopted in the present paper. The least action principle will be restated and the Hilbert action will still be used for the vacuum Einstein's equation. In order to show this is proper and natural, we will consider the variational principle for classical fields in  $n(\geq 2)$ -dimensional spacetime with a Lagrangian containing the field quantity and its derivatives of up to the  $N(\geq 1)$ -th order. In our opinion, non-local interaction is not acceptable, so we assume that the Lagrangian does not contain the integration of the field quantity.

There have been varied versions of variational principle and Noether's

theorem in the literature[5], and different notations have been used by different authors. For the readers' convenience, and for the consistency of the reasoning, we start with presenting a general variational principle for classical fields in  $n(\geq 2)$ -dimensional spacetime with a Lagrangian containing the field quantity and its derivatives of up to the  $N(\geq 1)$ -th order. In section 2, the Lagrangian formalism is presented, and a generalized Hamilton-Jacobi equation for Hamilton's principal functional is obtained. In section 3, the Noether's theorem is rederived. Then, in section 4, these general results are applied to the specific case of general relativity, especially Noether's theorem is applied to get quite a few conservation laws in GR. It is noted that most of the conserved quantities obtained here are not tensors like various gravitational energy-momentum complexes. Then Noether's theorem is revisited in section 5, and it is shown that the so-called pseudotensor problem of varied gravitational energy-momentum complexes is a result of taking different geometrical physical objects as one and the same; as a matter of fact, all the Noether's conserved quantities are scalars; and the non-locality difficulty does not really exist.

These results will be used to explore the energy-momentum conservation and the gravitational energy-momentum in GR in a later paper.

## 2 A general variational principle for classical fields

### 2.1 Lagrangian formulation

First, we present a useful mathematical formula for the variational principle, which does not rely on physics.

Suppose  $\{\Phi_B : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ (or } \mathbb{C}) \mid B = 1, 2, \dots, f\}$  ( $n \geq 2$ ) are smooth functions, and function  $L = L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x))$  is smooth with respect to all its arguments. It is easy to show just by using Leibniz's

rule that (the Einstein convention is used for coordinate indices)<sup>1</sup>

$$\begin{aligned}
\delta L &= \sum_{B=1}^f \sum_{X=0}^N \delta \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x) \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\
&= \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} + \partial_{\lambda} \sum_{B=1}^f \sum_{Z=0}^{N-1} \\
&\quad \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) \sum_{Y=0}^{N-1-Z} (-1)^Y \partial_{\mu_1} \cdots \partial_{\mu_Y} \frac{\partial L}{\partial \partial_{\lambda} \partial_{\mu_1} \cdots \partial_{\mu_Y} \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x)} \\
&=: \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\
&\quad + \partial_{\lambda} \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \cdots \nu_Z}(x)
\end{aligned} \tag{1}$$

when  $\Phi_B(x) \mapsto \tilde{\Phi}_B(x) = \Phi_B(x) + \delta \Phi_B(x), \forall B = 1, \dots, f$ . Consider the functional  $F$  of the following form

$$F[\Phi] = \int_{\Omega} d^n x L(x, \Phi(x), \partial \Phi(x), \partial^2 \Phi(x), \dots, \partial^N \Phi(x)), \tag{2}$$

where  $\Omega$  is an open subset with a compact closure of  $\mathbb{R}^n$ . When the arguments  $\Phi_B(x) \forall B = 1, \dots, f$  change slightly, the variation of functional  $F$  is

$$\delta F[\Phi] = \int_{\Omega} d^n x \sum_{B=1}^f \sum_{X=0}^N \delta \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x) \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)}$$

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<sup>1</sup>In order to avoid the indefiniteness of derivatives such as  $\frac{\partial}{\partial \partial_1 \partial_2 \Phi_B} L$ , and in order to keep the formulae neat, it is assumed in the present paper without loss of generality that

$$\begin{aligned}
&L(x, \Phi(x), \partial \Phi(x), \partial^2 \Phi(x), \dots, \partial^N \Phi(x)) \\
&= : L(x^\alpha, \Phi_B(x), \partial_{\alpha_1} \Phi_B(x), \partial_{\alpha_1} \partial_{\alpha_2} \Phi_B(x), \dots, \partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_N} \Phi_B(x)) \\
&= L(x^\alpha, \Phi_B(x), \partial_{\alpha} \Phi_B(x), \frac{1}{2!} \sum_{P \in S_2} \partial_{\alpha_{P(1)}} \partial_{\alpha_{P(2)}} \Phi_B(x), \dots, \\
&\quad \frac{1}{N!} \sum_{P \in S_N} \partial_{\alpha_{P(1)}} \partial_{\alpha_{P(2)}} \cdots \partial_{\alpha_{P(N)}} \Phi_B(x))
\end{aligned}$$

We will pretend that all the cross derivatives are independent variables of  $L$ , when calculating  $\frac{\partial}{\partial \partial_1 \partial_2 \Phi_B} L$ , etc. See Appendix A for the details.

$$\begin{aligned}
&= \int_{\Omega} d^n x \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\
&\quad + \int_{\Omega} d^n x \partial_{\lambda} \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \\
&= \int_{\Omega} d^n x \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\
&\quad + \int_{\partial\Omega} ds_{\lambda}(x) \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \tag{3}
\end{aligned}$$

This can be easily obtained by using eqn.(1) and the Stokes theorem. The derivatives of functional (2) is defined as follows.

**Definition 1** *If the change of the functional (2) can be expressed as*

$$F[\Phi + \delta\Phi] - F[\Phi] = \int_{\Omega} d^n x \sum_{B=1}^f \delta \Phi_B(x) D^B[\Phi, x] + o[\delta\Phi] \tag{4}$$

where  $D^B[\Phi, x]$  is a functional of  $\Phi$  varying with  $x$ , and  $o[\delta\Phi]$  is a higher order infinitesimal of  $\delta\Phi$ , when  $\{\Phi_B \mid B = 1, 2, \dots, f\}$  change slightly while the boundary values of  $\Phi, \partial\Phi, \dots, \partial^{N-1}\Phi$  are kept fixed, then  $F$  is called differentialble at  $\Phi$ , and  $D^B[\Phi, x]$  is called the derivative of functional  $F$  with respect to  $\Phi_B$  at  $\Phi$  and point  $x$ , and denoted by

$$D^B[\Phi, x] = \frac{\delta F[\Phi]}{\delta \Phi_B(x)} \tag{5}$$

Let us now apply the general formula (3) to the action functional of classical field  $\{\Phi_B : M \longrightarrow \mathbb{R} \text{ (or } \mathbb{C}) \mid B = 1, 2, \dots, f\}$

$$A[\Phi] = \int_{x(\Omega)} d^n x L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x)) \tag{6}$$

where  $\Omega$  is an open subset with a compact closure of the spacetime manifold  $M$ ,  $x(\Omega) \subset \mathbb{R}^n$  is the image of  $\Omega \subset M$  under the coordinate mapping  $x :$

$M \rightarrow \mathbb{R}^n$  and  $L$  is the Lagrangian of the field. We get the difference between the action functionals over  $\Omega$  of two possible movements (two paths allowed by the constraints) close to each other

$$\begin{aligned}
\delta A[\Phi] &= \int_{x(\Omega)} d^n x \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\
&\quad + \int_{x(\Omega)} d^n x \partial_\lambda \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \\
&= \int_{x(\Omega)} d^n x \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\
&\quad + \int_{x(\partial\Omega)} ds_\lambda(x) \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \tag{7}
\end{aligned}$$

Equation (7) suggests that for all  $N \geq 1$ ,  $n \geq 2$  **the least action principle** read as follows.

*For any spacetime region  $\Omega$ , among all possible movements in  $\Omega$  with the same boundary condition*

$$\delta \Phi|_{\partial\Omega} = 0, \delta \partial \Phi|_{\partial\Omega} = 0, \dots, \delta \partial^{N-1} \Phi|_{\partial\Omega} = 0, \tag{8}$$

*the real movement (the path allowed by physical laws) corresponds to the stationary value of the action over  $\Omega$ .*

Combining eqns.(7), (8), one obtains **the field equation (Euler-Lagrange equation)** satisfied by the real movements

$$\frac{\delta A[\Phi]}{\delta \Phi_B(x)} = \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} = 0. \tag{9}$$

This Lagrangian formalism applies to Newtonian dynamics, dynamics in special relativity (SR) and dynamics in general relativity. Specific covariance is the heritage from the Lagrangian being used. It is not the heritage from this general formalism.

## 2.2 Hamilton's principal functional and Hamilton-Jacobi's equation

Let us consider the difference between actions over spacetime region  $\Omega$  of two real movements close to each other. Using eqns.(7) and (9), one gets, for real movements

$$\delta A[\Phi] = \int_{x(\partial\Omega)} ds_\lambda(x) \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \quad (10)$$

From eqn.(10), one sees that the action over a spacetime region  $\Omega$  of a real movement is determined by the closed hyper-surface  $\partial\Omega$ , and  $\Phi|_{\partial\Omega}$ ,  $\partial\Phi|_{\partial\Omega}, \dots, \partial^{N-1}\Phi|_{\partial\Omega}$ . It will be called the generalized Hamilton's principal functional and denoted by

$$S = S[\partial\Omega, \Phi|_{\partial\Omega}, \partial\Phi|_{\partial\Omega}, \dots, \partial^{N-1}\Phi|_{\partial\Omega}] \quad (11)$$

Re-write eqn.(10) as

$$\delta S = \int_{x(\partial\Omega)} ds_\lambda(x) \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \quad (12)$$

Note that when  $\Phi_B|_{\partial\Omega}$  is given, only one of the  $n$  derivatives  $\partial_\lambda \Phi_B|_{\partial\Omega}$  ( $\lambda = 0, 1, \dots, n-1$ ) is independent; when  $\partial_\lambda \Phi_B|_{\partial\Omega}$  is given, only one of the  $n$  derivatives  $\partial_\mu \partial_\lambda \Phi_B|_{\partial\Omega}$  ( $\mu = 0, 1, \dots, n-1$ ) is independent; and so on. Thus for a given suffix  $B$ , only  $N$  items (  $\Phi_B|_{\partial\Omega}$ , one of  $\partial_{\lambda_1} \Phi_B|_{\partial\Omega}, \dots$ , one of  $\partial_{\lambda_1} \cdots \partial_{\lambda_{N-1}} \Phi_B|_{\partial\Omega}$  ( $\lambda_j = 0, 1, \dots, n-1$ ) ) are independent.

In order to formulate the generalized Hamilton-Jacobi's equation, one needs a new type of functional derivative[6].

**Definition 2** Let  $\Sigma$  be a closed hypersurface in  $\mathbb{R}^n$ ,  $\Psi$  a function defined on  $\mathbb{R}^n$ , and  $F = F[\Sigma, \Psi|_\Sigma]$  a functional of  $\Sigma$  and  $\Psi|_\Sigma$ . The functional derivatives are defined as follows. If the variation of  $F$  can be written as

$$\delta F[\Sigma, \Psi|_\Sigma] = \int_\Sigma ds_\lambda(x) \{ Y[\Sigma, \Psi|_\Sigma, x]^\lambda_\mu \delta \Sigma^\mu(x) + Z[\Sigma, \Psi|_\Sigma, x]^\lambda \delta \Psi(x) \} \quad (13)$$

where  $Y[\Sigma, \Psi|_\Sigma, x]^\lambda_\mu$  and  $Z[\Sigma, \Psi|_\Sigma, x]^\lambda$  are functionals of  $\Sigma$  and  $\Psi|_\Sigma$ , which vary with  $x$ , when  $\Sigma \mapsto \tilde{\Sigma}$ , and  $\Psi \mapsto \tilde{\Psi}$ , then they are called the functional

derivative of  $F$  with respect to  $\Sigma^\mu(x)$  and  $\Psi(x)$ , and denoted by

$$Y[\Sigma, \Psi|_\Sigma, x]_\mu^\lambda = \left( \frac{\delta F}{\delta \Sigma^\mu(x)} \right)^\lambda, \quad Z[\Sigma, \Psi|_\Sigma, x]^\lambda = \left( \frac{\delta F}{\delta \Psi(x)} \right)^\lambda \quad (14)$$

respectively.

Hence we have

$$\delta F[\Sigma, \Psi|_\Sigma] = \int_\Sigma ds_\lambda(x) \left[ \left( \frac{\delta F}{\delta \Sigma^\mu(x)} \right)^\lambda \delta \Sigma^\mu(x) + \left( \frac{\delta F}{\delta \Psi(x)} \right)^\lambda \delta \Psi(x) \right] \quad (15)$$

The hypersurface  $\Sigma$  is given by the parameter equation

$$x^\mu = \Sigma^\mu(\theta^1, \theta^2, \theta^3) \quad (16)$$

The  $\delta \Sigma^\mu(x)$  and  $\delta \Psi(x)$  in eqn.(13) are respectively

$$\begin{aligned} \delta \Sigma^\mu(x) &= \tilde{\Sigma}^\mu(\theta^1, \theta^2, \theta^3) - \Sigma^\mu(\theta^1, \theta^2, \theta^3), \\ \delta \Psi(x) &= \tilde{\Psi}(\tilde{\Sigma}^\mu(\theta^1, \theta^2, \theta^3)) - \Psi(\Sigma^\mu(\theta^1, \theta^2, \theta^3)). \end{aligned} \quad (17)$$

Now, from eqns.(12) and (15) we get  $[\partial \Omega$  in (12) is  $\Sigma$  in (15)]

$$\begin{aligned} \left( \frac{\delta S}{\delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x)} \right)^\lambda &= K^{B\lambda\nu_1 \dots \nu_Z}(x), \\ \forall Z &= 0, \dots, N-1, B = 1, \dots, f. \end{aligned} \quad (18)$$

Following the evolution of one real movement and observing the change of its action with  $\Sigma$ ,

$$\begin{aligned} \delta S &= \int_{x(\partial \Omega)} ds_\lambda(x) L \delta_\sigma^\lambda \delta \Sigma^\sigma(x) \\ &= \int_{x(\partial \Omega)} ds_\lambda(x) \left[ \left( \frac{\delta S}{\delta \Sigma^\sigma(x)} \right)^\lambda \delta \Sigma^\sigma(x) \right. \\ &\quad \left. + \sum_{B=0}^f \sum_{Z=0}^{N-1} \left( \frac{\delta S}{\delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x)} \right)^\lambda \partial_\sigma \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) \delta \Sigma^\sigma(x) \right] \end{aligned} \quad (19)$$



We get the generalized Hamilton-Jacobi's equation.

$$\left(\frac{\delta S}{\delta \Sigma^\sigma(x)}\right)^\lambda + [L\delta_\sigma^\lambda - \sum_{B=0}^f \sum_{Z=0}^{N-1} \partial_\sigma \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x)] = 0 \quad (20)$$

It will be shown in the next section that the expression in the bracket is just the canonical energy-momentum  $\tau_\sigma^\lambda(x)$ , hence we have

$$\left(\frac{\delta S}{\delta \Sigma^\sigma(x)}\right)^\lambda + \tau_\sigma^\lambda(x) = 0. \quad (21)$$

For a dynamic system with finite degrees of freedom, we have the Hamilton's principal function  $S(q^1, \dots, q^h, t)$ , and the Hamilton-Jacobi's equation is a partial differential equation  $\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0$ . While for the classical field, we have the Hamilton's principal functional, and the Hamilton-Jacobi's equation is a set of differential equations for functional. It plays an important role in canonical quantization for fields.

## 3 Noether's theorem

### 3.1 Re-deriving the theorem

There have been varied versions of Noether's theorem, and different notations have been used. To avoid confusion, here we redrive Noether's theorem for classical fields in  $n$ -dimensional spacetime with a Lagrangian containing field quantity and its derivatives of up to the  $N$ -th order, in terms of coordinate language.

**Theorem 3** *If the action of classical fields over every spacetime region  $\Omega$  with a compact closure*

$$A[\Phi] = \int_{x(\Omega)} d^n x L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x))$$

*remains unchanged under the following  $r$ -parameter family of infinitesimal*

*transformations of coordinates and field quantities*

$$\begin{aligned}
x^\lambda &\longmapsto \tilde{x}^\lambda = x^\lambda + \delta x^\lambda, \\
\delta x^\lambda &=: \delta x^\lambda(x, \epsilon^1, \dots, \epsilon^r), |\epsilon^i| \ll 1, \delta x^\lambda(x, 0, \dots, 0) = 0 \\
\Phi_B(x) &\longmapsto \tilde{\Phi}_B(\tilde{x}) = \Phi_B(x) + \delta\Phi_B(x), \\
\delta\Phi_B(x) &=: \delta\Phi_B(x, \epsilon^1, \dots, \epsilon^r), |\epsilon^i| \ll 1, \delta\Phi_B(x, 0, \dots, 0) = 0 \quad (22)
\end{aligned}$$

*then there exist  $r$  conservation laws.*

**Proof.** The small change of field quantity  $\delta\Phi_B(x)$  can be divided into two parts, the part due to the small change of its function form and the part due to the small change of its arguments respectively.

$$\delta\Phi_B(x) = \bar{\delta}\Phi_B(x) + \delta x^\sigma \partial_\sigma \Phi_B(x) \quad (23)$$

Similarly the small change of derivatives of field quantity  $\delta[\partial_{\lambda_1} \partial_{\lambda_2} \cdots \partial_{\lambda_X} \Phi_B(x)]$  can be written as

$$\begin{aligned}
\delta[\partial_{\lambda_1} \partial_{\lambda_2} \cdots \partial_{\lambda_X} \Phi_B(x)] &= \partial_{\lambda_1} \partial_{\lambda_2} \cdots \partial_{\lambda_X} \bar{\delta}\Phi_B(x) \\
&\quad + \delta x^\sigma \partial_\sigma \partial_{\lambda_1} \partial_{\lambda_2} \cdots \partial_{\lambda_X} \Phi_B(x) \quad (24)
\end{aligned}$$

The variation of the action can be divided into two parts. One is due to the small change of the integration domain  $x(\Omega) \longmapsto \tilde{x}(\Omega)$  in  $\mathbb{R}^n$ , and the other is due to the small change of the integrand

$$\begin{aligned}
&L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x)) \\
&\longmapsto L(x, \tilde{\Phi}(x), \partial\tilde{\Phi}(x), \partial^2\tilde{\Phi}(x), \dots, \partial^N\tilde{\Phi}(x))
\end{aligned}$$

$$\begin{aligned}
\delta A[\Phi] &= \int_{x(\partial\Omega)} ds_\lambda(x) \delta x^\lambda L \\
&\quad + \int_{x(\Omega)} d^n x \sum_{B=1}^f \sum_{X=0}^N \frac{\partial L}{\partial \partial_{\mu_1} \cdots \partial_{\mu_X} \Phi_B(x)} \partial_{\mu_1} \cdots \partial_{\mu_X} \bar{\delta}\Phi_B(x) \\
&= \int_{x(\Omega)} d^n x \sum_{B=1}^f \bar{\delta}\Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\mu_1} \cdots \partial_{\mu_X} \frac{\partial L}{\partial \partial_{\mu_1} \cdots \partial_{\mu_X} \Phi_B(x)} \\
&\quad + \int_{x(\Omega)} d^n x \partial_\lambda [\delta x^\sigma \delta_\sigma^\lambda L + \sum_{B=1}^f \sum_{Z=0}^{N-1} \partial_{\nu_1} \cdots \partial_{\nu_Z} \bar{\delta}\Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x)] \quad (25)
\end{aligned}$$

The first integrand at rhs vanishes for real movement, hence the second integral does too. One gets the following continuity equation due to the arbitrariness of  $\Omega$ .

$$\partial_\lambda [\delta x^\sigma \delta_\sigma^\lambda L + \sum_{B=1}^f \sum_{Z=0}^{N-1} \partial_{\nu_1} \cdots \partial_{\nu_Z} (\delta \Phi_B(x) - \delta x^\sigma \partial_\sigma \Phi_B(x)) K^{B\lambda\nu_1 \dots \nu_Z}(x)] = 0$$

or

$$\int_{x(\partial\Omega)} ds_\lambda(x) \left\{ \delta x^\sigma \delta_\sigma^\lambda L + \sum_{B=1}^f \sum_{Z=0}^{N-1} [\partial_{\nu_1} \cdots \partial_{\nu_Z} (\delta \Phi_B(x) - \delta x^\sigma \partial_\sigma \Phi_B(x)) K^{B\lambda\nu_1 \dots \nu_Z}(x)] \right\} = 0 \quad (26)$$

Noting that both  $\delta x^\sigma$  and  $\delta \Phi_B(x)$  depend on  $r$  real parameters, one gets from eqn.(26)  $r$  conservation laws.

$$\begin{aligned} \partial_\lambda \left\{ \sum_{B=1}^f \sum_{Z=0}^{N-1} \partial_{\nu_1} \cdots \partial_{\nu_Z} \left( \frac{\partial \delta \Phi_B(x)}{\partial \epsilon^\alpha} \Big|_{\epsilon=0} - \frac{\partial \delta x^\sigma}{\partial \epsilon^\alpha} \Big|_{\epsilon=0} \partial_\sigma \Phi_B(x) \right) K^{B\lambda\nu_1 \dots \nu_Z}(x) \right. \\ \left. + \frac{\partial \delta x^\sigma}{\partial \epsilon^\alpha} \Big|_{\epsilon=0} \delta_\sigma^\lambda L \right\} = 0, \quad \forall \alpha = 1, \dots, r \end{aligned} \quad (27)$$

■

## 3.2 Conservation law due to “coordinate shift” invariance

In this subsection, we restrict our discussion to those cases, such that the Lagrangian does not manifestly contain coordinates and the action of classical field over any spacetime region

$$A[\Phi] = \int_{x(\Omega)} d^n x L(\Phi(x), \partial \Phi(x), \partial^2 \Phi(x), \dots, \partial^N \Phi(x)) \quad (28)$$

remains unchanged under the following “coordinate shift”.

$$\delta x^\sigma = \varepsilon^\sigma, \quad \delta \Phi_B(x) = 0. \quad (29)$$

And eqn.(27) reads

$$\partial_\lambda \tau_\sigma^\lambda(x) = 0, \quad (30)$$

where

$$\tau_\sigma^\lambda(x) = \delta_\sigma^\lambda L - \sum_{B=1}^f \sum_{Z=0}^{N-1} \partial_\sigma \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \quad (31)$$

is usually called canonical energy-momentum.

Comparing eqn.(20) and (31), we get eqn.(21), the generalized Hamilton-Jacobi equation.

The formalism presented so far is good for any classical field in  $n$ -dimensional spacetime with a Lagrangian containing field quantities and their derivatives of up to the  $N$ -th order, no matter it is Galileo covariant, Lorentz covariant, general covariant or without any covariance. Therefore, we can not say eqn.(21) is covariant even though it looks like so. The covariance of the formalism is the heritage from the Lagrangian being used, not the heritage from the formalism itself.

## 4 Variational principle approach to general relativity

Let us apply the general results obtained above to the classic fields in GR. We will consider the case where the matter field is a  $(1,1)$ -tensor field. The results can be readily generalized to any  $(r,s)$ -tensor matter field. For the dynamic system,  $(1,1)$ -tensor matter field  $u_\xi^\theta(x)$  plus the metric field  $g_{\alpha\beta}(x)$ , the Lagrangian and the action over any spacetime region  $\Omega$  are

$$\begin{aligned} & L(g(x), \partial g(x), \partial^2 g(x), u(x), \partial u(x)) \\ = & \sqrt{-|g(x)|} [\mathcal{L}(g(x), u(x), \nabla u(x)) + \frac{1}{16\pi G} R] = L_M + L_G, \end{aligned} \quad (32)$$

$$\begin{aligned} A[g, u] = & \int_{x(\Omega)} d^4x \sqrt{-|g(x)|} [\mathcal{L}(g(x), u(x), \nabla u(x)) \\ & + \frac{1}{16\pi G} R] = A_M[g, u] + A_G[g] \end{aligned} \quad (33)$$

where  $R$  is the Ricci scalar curvature,  $\mathcal{L}(g(x), u(x), \nabla u(x))$  is the Lagrangian for matter field obtained from the Lagrangian in special relativity

$\mathcal{L}(\eta, u(x), \partial u(x))$  by replacing the Minkowski metric  $\eta = \text{diag}(-1, 1, 1, 1)$  with  $g_{\alpha\beta}(x)$ , and replacing the partial derivatives  $\partial_\mu u_\xi^\theta(x)$  with the covariant derivatives  $\nabla_\mu u_\xi^\theta(x)$ .

## 4.1 Einstein's field equation

The Euler-Lagrange equation, Eqn.(9) now reads

$$\frac{\delta A[g, u]}{\delta u_\varphi^\theta(x)} = \sqrt{-|g(x)|} \left[ \frac{\partial \mathcal{L}}{\partial u_\varphi^\theta(x)} - \nabla_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\varphi^\theta(x)} \right] = 0 \quad (34)$$

$$\frac{\delta A[g, u]}{\delta g_{\alpha\beta}(x)} = \sqrt{-|g(x)|} \frac{1}{16\pi G} [R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}(x) - 8\pi G T^{\alpha\beta}] = 0 \quad (35)$$

where  $T^{\alpha\beta}$  is the energy-momentum tensor of matter field, which is a symmetrical (2,0)-tensor.

$$\begin{aligned} T^{\alpha\beta} &\equiv \frac{2}{\sqrt{-|g(x)|}} \frac{\delta A_M[g, u]}{\delta g_{\alpha\beta}(x)} = g^{\alpha\beta}(x) \mathcal{L} + 2 \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}(x)} \Big|_{u(x), \nabla u(x)} \\ &+ 2 \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\xi^\theta(x)} \frac{\partial \nabla_\lambda u_\xi^\theta(x)}{\partial g_{\alpha\beta}(x)} - 2 \Gamma_{\nu\lambda}^\nu(x) \frac{\partial \mathcal{L}}{\partial \nabla_\mu u_\xi^\theta(x)} \frac{\partial \nabla_\mu u_\xi^\theta(x)}{\partial \partial_\lambda g_{\alpha\beta}(x)} \\ &- 2 \partial_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\mu u_\xi^\theta(x)} \frac{\partial \nabla_\mu u_\xi^\theta(x)}{\partial \partial_\lambda g_{\alpha\beta}(x)} - 2 \frac{\partial \mathcal{L}}{\partial \nabla_\mu u_\xi^\theta(x)} \partial_\lambda \frac{\partial \nabla_\mu u_\xi^\theta(x)}{\partial \partial_\lambda g_{\alpha\beta}(x)} \\ &= T^{\alpha\beta}(u(x), \partial u(x), g(x), \partial g(x)) \end{aligned} \quad (36)$$

All the pre-GR dynamics discuss how matter behaves in spacetime with a given metric, that is, Newton's absolute space-time or Minkowski space. All of them can not explain why there are inertial reference systems and non-inertial reference systems, and why the inertial mass equals the gravitational mass. G.R. is unique. It discusses how the matter motion determines the bending of the spacetime and how the matter behaves in a curved spacetime. Hence the dynamical variables are both the matter field  $u_\xi^\theta(x)$  and the metric field of spacetime  $g_{\alpha\beta}(x)$ . It means that they are both to be determined simultaneously by solving the equation of motion. It does not necessarily imply that the metric field of spacetime  $g_{\alpha\beta}(x)$  is matter (in its philosophic context, in the most general meaning of the word) like, say, the electromagnetic field, the spinor field, etc. It does not necessarily imply that the metric field of spacetime  $g_{\alpha\beta}(x)$  carries energy-momentum.

## 4.2 Noether's theorem for classical field in GR

The Noether's conservation law, or the continuity equation (26) , now reads

$$\frac{\partial}{\partial x^\kappa} \left\{ \sqrt{-|g(x)|} J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)] \right\} = 0$$

or

$$\int_{x(\partial\Omega)} ds_\lambda(x) \{ \sqrt{-|g(x)|} J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)] \} = 0 \quad (37)$$

where

$$\begin{aligned} J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)] &=: J_x^\kappa \\ &= \{ \mathcal{L} \delta_\rho^\kappa \delta x^\rho + \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} \bar{\delta} u_\xi^\theta(x) + \frac{1}{2} [ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) \\ &+ \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\ &- \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) ] \bar{\delta} g_{\alpha\beta}(x) \} + \\ &\frac{1}{16\pi G} \{ R \delta_\rho^\kappa \delta x^\rho + [ \frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \\ &- \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} ] \bar{\delta} g_{\alpha\beta}(x) + \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu \bar{\delta} g_{\alpha\beta}(x) \} \end{aligned} \quad (38)$$

It is worth noting that the form of function  $J^\kappa$  is independent of coordinate systems, and the arguments of function  $J^\kappa$  are  $u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x);$  and  $\delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)$ .

## 4.3 Conservation laws in GR

### 4.3.1 Conservation law due to “coordinate shift” invariance

Action (33) remains unchanged under the following “coordinate shifts”.

$$\delta x^\rho = \epsilon^\rho, \quad \delta u_\xi^\theta(x) = 0, \quad \delta g_{\alpha\beta}(x) = 0. \quad (39)$$

In this case, eqn.(26) reads

$$\partial_\lambda[\sqrt{-|g(x)|}\tau_\rho^\lambda(x)] = 0, \quad (40)$$

where

$$\begin{aligned} \tau_\rho^\kappa(x) &= \tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \\ &= \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} \partial_\rho u_\xi^\theta(x) - \mathcal{L} \delta_\rho^\kappa + \frac{1}{2} \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) \right. \\ &\quad + \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) \\ &\quad - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\ &\quad + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) \left. \right] \partial_\rho g_{\alpha\beta}(x) + \frac{1}{16\pi G} \\ &\quad \left[ \left( \frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \right) \partial_\rho g_{\alpha\beta}(x) \right. \\ &\quad \left. + \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu \partial_\rho g_{\alpha\beta}(x) - R \delta_\rho^\kappa \right] \end{aligned} \quad (41)$$

is usually called canonical energy-momentum tensor. It is worth noting that the arguments of function  $\tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$  are all **field quantities and their derivatives**. Notice that "coordinate shift" eqn.(39) is not an invariant concept under general coordinate transformation. This explains why  $\tau_\sigma^\lambda(x)$  is not a tensor under general coordinate transformation. We will get back to this problem later.

#### 4.3.2 Conservation law due to "4-dimensional rotation" invariance

The action (33) remains unchanged under infinitesimal "4-dimensional rotations" (Lorentz transformations), which form a 6-parameter family of infinitesimal symmetry transformations

$$\begin{aligned} x^\mu &\longmapsto \tilde{x}^\mu = L_\nu^\mu x^\nu, \quad L_\nu^\mu = \delta_\nu^\mu + \Lambda_\nu^\mu, \quad |\Lambda_\nu^\mu| \ll 1, \quad \eta_{\mu\lambda} \Lambda_\nu^\lambda \equiv \Lambda_{\mu\nu}, \quad \Lambda_{\mu\nu} = -\Lambda_{\nu\mu}, \\ \delta x^\lambda &= \Lambda_\mu^\lambda x^\mu = \frac{1}{2} (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \Lambda_{\rho\sigma}, \end{aligned} \quad (42)$$

$$\begin{aligned}\delta u_\xi^\theta(x) &= \Lambda_\varphi^\theta u_\xi^\varphi(x) - \Lambda_\xi^\eta u_\eta^\theta(x) \\ &= \Lambda_{\rho\sigma} \frac{1}{2} [\eta^{\theta\rho} u_\xi^\sigma(x) - \eta^{\theta\sigma} u_\xi^\rho(x) - \delta_\xi^\sigma \eta^{\eta\rho} u_\eta^\theta(x) + \delta_\xi^\rho \eta^{\eta\sigma} u_\eta^\theta(x)],\end{aligned}\quad (43)$$

$$\begin{aligned}\delta g_{\alpha\beta}(x) &= \Lambda_{\rho\sigma} \frac{1}{2} [-\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) \\ &\quad + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) - \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)]\end{aligned}\quad (44)$$

In this case eqn.(26) reads

$$\frac{\partial}{\partial x^\kappa} \{ \sqrt{-|g(x)|} M^{\kappa\rho\sigma} \} = 0 \quad (45)$$

where

$$\begin{aligned}& 2M^{\kappa\rho\sigma}(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \\ &= \mathcal{L} \delta_\lambda^\kappa (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) + \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} \{ [\eta^{\theta\rho} u_\xi^\sigma(x) - \eta^{\theta\sigma} u_\xi^\rho(x) \\ &\quad - \delta_\xi^\sigma \eta^{\eta\rho} u_\eta^\theta(x) + \delta_\xi^\rho \eta^{\eta\sigma} u_\eta^\theta(x)] - (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \partial_\lambda u_\xi^\theta(x) \} \\ &\quad + \frac{1}{2} \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) \right. \\ &\quad - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\ &\quad \left. - \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) \right] \times \\ &\quad \{ [-\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) - \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \\ &\quad - (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x) \} + \frac{1}{16\pi G} \{ R \delta_\lambda^\kappa (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \\ &\quad + \left( \frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \right) \\ &\quad ([-\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) - \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \\ &\quad - (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x)) + \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu ([-\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) \\ &\quad + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) - \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \\ &\quad - (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x)) \} \end{aligned}\quad (46)$$



### 4.3.3 Conservation law due to “4-dimensional pure deformation” invariance

The action (33) remains unchanged under infinitesimal “4-dimensional pure deformations”, which form a 6-parameter family of infinitesimal symmetry transformations

$$\begin{aligned} x^\mu &\longmapsto \tilde{x}^\mu = L_\nu^\mu x^\nu, \quad L_\nu^\mu = \delta_\nu^\mu + \Lambda_\nu^\mu, \quad |\Lambda_\nu^\mu| \ll 1, \\ \eta_{\mu\lambda} \Lambda_\nu^\lambda &\equiv \Lambda_{\mu\nu}, \quad \Lambda_{\mu\nu} = \Lambda_{\nu\mu}, \end{aligned} \quad (47)$$

$$\delta x^\lambda = \Lambda_\mu^\lambda x^\mu = \frac{1}{2}(\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \Lambda_{\rho\sigma}, \quad (48)$$

$$\begin{aligned} \delta u_\xi^\theta(x) &= \Lambda_\varphi^\theta u_\xi^\varphi(x) - \Lambda_\xi^\eta u_\eta^\theta(x) \\ &= \frac{1}{2}[\eta^{\theta\rho} u_\xi^\sigma(x) + \eta^{\theta\sigma} u_\xi^\rho(x) - \delta_\xi^\sigma \eta^{\rho\sigma} u_\eta^\theta(x) - \delta_\xi^\rho \eta^{\eta\sigma} u_\eta^\theta(x)] \Lambda_{\rho\sigma}, \end{aligned} \quad (49)$$

$$\begin{aligned} \delta g_{\alpha\beta}(x) &= -\frac{1}{2}[\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) \\ &\quad + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) + \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \Lambda_{\rho\sigma} \end{aligned} \quad (50)$$

In this case eqn.(26) reads

$$\frac{\partial}{\partial x^\kappa} \{ \sqrt{-|g(x)|} N^{\kappa\rho\sigma} \} = 0 \quad (51)$$

where

$$\begin{aligned} &2N^{\kappa\rho\sigma}(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \\ &= \mathcal{L} \delta_\lambda^\kappa (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) + \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} \{ [\eta^{\theta\rho} u_\xi^\sigma(x) + \eta^{\theta\sigma} u_\xi^\rho(x) \\ &\quad - \delta_\xi^\sigma \eta^{\eta\rho} u_\eta^\theta(x) - \delta_\xi^\rho \eta^{\eta\sigma} u_\eta^\theta(x)] - (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \partial_\lambda u_\xi^\theta(x) \} \\ &\quad + \frac{1}{2} \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) \right. \\ &\quad - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\ &\quad \left. - \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) \right] \times \\ &\quad \{ [\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) + \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \} \end{aligned}$$

$$\begin{aligned}
& + (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x) \} + \frac{1}{16\pi G} \{ R \delta_\lambda^\kappa (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \\
& - (\frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)}) \\
& [\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) + \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) \\
& + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x) + (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x)] \\
& - \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu [\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) + \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) \\
& + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x) + (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x)] \} \quad (52)
\end{aligned}$$

#### 4.3.4 Conservation law due to “scaling” invariance

The action (33) remains unchanged under infinitesimal scaling transformations, which form a 1-parameter family of infinitesimal symmetry transformations

$$x^\lambda \mapsto \tilde{x}^\lambda = e^\epsilon x^\lambda, \quad |\epsilon| \ll 1, \quad \delta x^\lambda = \epsilon x^\lambda, \quad \forall \lambda = 0, 1, 2, 3 \quad (53)$$

$$\delta u_\xi^\theta(x) = 0, \quad \delta g_{\alpha\beta}(x) = -2\epsilon g_{\alpha\beta}(x) \quad (54)$$

In this case eqn.(26) reads

$$\frac{\partial}{\partial x^\kappa} \{ \sqrt{-|g(x)|} S^\kappa \} = 0 \quad (55)$$

$$\begin{aligned}
& S^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \\
& = \mathcal{L} \delta_\rho^\kappa x^\rho - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} x^\rho \frac{\partial}{\partial x^\rho} u_\xi^\theta(x) - \frac{1}{2} [ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) \\
& + \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\
& - \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) ] [2g_{\alpha\beta}(x) + x^\rho \partial_\rho g_{\alpha\beta}(x)] \\
& + \frac{1}{16\pi G} \{ R \delta_\rho^\kappa x^\rho - [ \frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} ] \\
& [2g_{\alpha\beta}(x) + x^\rho \partial_\rho g_{\alpha\beta}(x)] - \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu [2g_{\alpha\beta}(x) + x^\rho \partial_\rho g_{\alpha\beta}(x)] \} \quad (56)
\end{aligned}$$

#### 4.3.5 Conservation law due to “skew-scaling” invariance

The action (33) remains unchanged under infinitesimal “skew-scaling” transformations, which form a 1-parameter family of infinitesimal symmetry transformations

$$\delta x^0 = -\epsilon^1 x^0, \quad \delta x^1 = \epsilon^1 x^1, \quad \delta x^2 = 0, \quad \delta x^3 = 0 \quad (57)$$

$$\left[ \frac{\partial \tilde{x}}{\partial x} \right] = \begin{bmatrix} 1 - \epsilon^1 & 0 & 0 & 0 \\ 0 & 1 + \epsilon^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (58)$$

$$\begin{aligned} \delta u_0^0(x) &= 0, \quad \delta u_1^0(x) = -2\epsilon^1 u_1^0(x), \quad \delta u_2^0(x) = -\epsilon^1 u_2^0(x), \quad \delta u_3^0(x) = -\epsilon^1 u_3^0(x) \\ \delta u_0^1(x) &= 2\epsilon^1 u_0^1(x), \quad \delta u_1^1(x) = 0, \quad \delta u_2^1(x) = \epsilon^1 u_2^1(x), \quad \delta u_3^1(x) = \epsilon^1 u_3^1(x) \\ \delta u_0^j(x) &= \epsilon^1 u_0^j(x), \quad \delta u_1^j(x) = -\epsilon^1 u_1^j(x), \quad \delta u_k^j(x) = 0, \quad \forall j, k = 2, 3 \\ \delta g_{00}(x) &= 2\epsilon^1 g_{00}(x), \quad \delta g_{01}(x) = 0, \quad \delta g_{02}(x) = \epsilon^1 g_{02}(x), \quad \delta g_{03}(x) = \epsilon^1 g_{03}(x) \\ \delta g_{10}(x) &= 0, \quad \delta g_{11}(x) = -2\epsilon^1 g_{11}(x), \quad \delta g_{12}(x) = -\epsilon^1 g_{12}(x), \quad \delta g_{13}(x) = -\epsilon^1 g_{13}(x) \\ \delta g_{j0}(x) &= \epsilon^1 g_{j0}(x), \quad \delta g_{j1}(x) = -\epsilon^1 g_{j1}(x), \quad \delta g_{jk}(x) = 0, \quad \forall j, k = 2, 3 \end{aligned} \quad (59)$$

Substitute eqns.(57) and (59) into eqn.(26), we can get a conserved current

$$\frac{\partial}{\partial x^\kappa} \{ \sqrt{-|g(x)|} J_1^\kappa \} = 0 \quad (60)$$

Here we skip the expression of  $J_1^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$ . Similarly we can get conserved currents  $J_2^\kappa$  and  $J_3^\kappa$ .

Because the symmetry group of classical field in GR is an infinite dimensional Lie group, we can get infinitely many conservation laws by using Noether’s theorem. It is interesting to note that most of the conserved currents are not tensors. For instance,  $\tau_\rho^\kappa$  is not a (1,1) tensor,  $M^{\kappa\rho\sigma}$  is not a (3,0) tensor,  $N^{\kappa\rho\sigma}$  is not a (3,0) tensor, and  $S^\kappa$  is not a (1,0) tensor, etc. One of the main tasks of the present work is to show that these pseudotensor problems (including the pseudotensor problem of various gravitational energy-momentum complexes) are just results from misreading Noether’s theorem.

## 5 Noether's theorem revisited

Let us start with the simplest example. The canonical energy-momentum  $\tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$  in eqn.(41) is not a tensor

$$\tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \neq \frac{\partial x^\kappa}{\partial y^\lambda} \frac{\partial y^\sigma}{\partial x^\rho} \tau_\sigma^\lambda(u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y)) . \quad (61)$$

This can be understood as follows. The 4-parameter families of infinitesimal transformations of coordinates and field quantities eqn.(39) and

$$\delta y^\rho = \epsilon^\rho, \quad \delta u_\xi^\theta(y) = 0, \quad \delta g_{\alpha\beta}(y) = 0. \quad (39')$$

do not correspond to the same 4-parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$ , when

$$\exists \quad p \in \mathbb{M}, \text{ and } 0 \leq \alpha, \beta, \mu \leq 3, \text{ such that } \frac{\partial^2 y^\mu}{\partial x^\alpha \partial x^\beta} \big|_p \neq 0, \quad (62)$$

Therefore, generally speaking, the conservation laws (continuity equations) corresponding to them are not equivalent to each other, and the conserved quantities corresponding to them are not the same geometrical physical object. Inequality (61) is a comparison between components of two different geometrical physical objects which have been mistaken one and the same.

In general, infinitesimal transformations of coordinates and field quantities expressed in different coordinate systems with the same form (like eqns.(39) and (39')) do not correspond to the same infinitesimal diffeomorphism of spacetime  $\mathbb{M}$ . Conversely, the same r-parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$  is described differently in different coordinate systems. For instance, if a 4-parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$  is described in coordinate system  $(x^0, x^1, x^2, x^3)$  by eqn.(39), then it would be described in coordinate system  $(y^0, y^1, y^2, y^3)$  not by eqn.(39'), but by

$$\begin{aligned} \delta y^\rho &= \frac{\partial y^\rho}{\partial x^\mu} \epsilon^\mu, \quad \delta u_\xi^\theta(y) = \epsilon^\mu \left[ \frac{\partial}{\partial y^\varphi} \left( \frac{\partial y^\theta}{\partial x^\mu} \right) u_\xi^\varphi(x) - \frac{\partial}{\partial y^\xi} \left( \frac{\partial y^\theta}{\partial x^\mu} \right) u_\eta^\theta(x) \right] \\ \delta g_{\alpha\beta}(y) &= -\epsilon^\mu \left[ \frac{\partial}{\partial y^\beta} \left( \frac{\partial y^\sigma}{\partial x^\mu} \right) g_{\alpha\sigma}(x) + \frac{\partial}{\partial y^\alpha} \left( \frac{\partial y^\rho}{\partial x^\mu} \right) g_{\rho\beta}(x) \right] \end{aligned} \quad (39'')$$

which generally is no longer “coordinate shift”.

It is generally accepted that in GR, the same physical geometrical object is expressed in different coordinate systems by functions of **field quantities, their derivatives, and coordinates**, with the same function form. But this is not always true. For instance, the Noether’s conserved current (38)  $J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)]$ , and its counterpart corresponding to the same  $r$ -parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$ , but written in coordinate system  $(y^0, y^1, y^2, y^3)$ ,  $J^\kappa [u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y); \delta y, \bar{\delta} u(y), \bar{\delta} g(y), \partial \bar{\delta} g(y)]$ , are not function of field quantities, their derivatives, and coordinates, with the same function form. Because  $J^\kappa$  manifestly contains  $\delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)$ , besides field quantities, their derivatives and coordinates, while  $\delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)$  and  $\delta y, \bar{\delta} u(y), \bar{\delta} g(y), \partial \bar{\delta} g(y)$  are different functions of field quantities, and their derivatives, and coordinates. Yet they are the same physical geometrical object, as will be shown in the following.

To prove this, let us show that continuity equations written in different coordinate systems, but corresponding to the same  $r$ -parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$ , are equivalent to each other, even though the conserved currents written in different coordinate systems are different functions of matter field, metric field, their derivatives, and coordinates.

Let  $\{\Delta(\epsilon^1, \dots, \epsilon^r) =: \Delta : \mathbb{M} \rightarrow \mathbb{M} \mid |\epsilon^i| \ll 1, \forall 1 \leq i \leq r\}$  be an  $r$ -parameter family of infinitesimal diffeomorphisms of spacetime manifold  $\mathbb{M}$  onto itself, and  $\Delta(0, \dots, 0)$  the identity mapping of  $\mathbb{M}$ . For any chart  $(U, \varphi)$  of  $\mathbb{M}$ , denoting by  $(x^0, x^1, x^2, x^3)$  the corresponding coordinate system, we have

$$(x^0, x^1, x^2, x^3) \xrightarrow{\pi^\lambda \circ \varphi \circ \Delta \circ \varphi^{-1}} \tilde{x}^\lambda = x^\lambda + \delta x^\lambda(x; \epsilon^1, \dots, \epsilon^r) =: x^\lambda + \delta x^\lambda, \quad (63)$$

where  $\pi^\lambda : (x^0, x^1, x^2, x^3) \mapsto x^\lambda$  is the projection operator,

$$\begin{aligned} \delta u_\xi^\theta(x; \epsilon^1, \dots, \epsilon^r) &= : \delta u_\xi^\theta(x) = \frac{\partial \delta x^\theta}{\partial x^\varphi} u_\xi^\varphi(x) - \frac{\partial \delta x^\eta}{\partial x^\xi} u_\eta^\theta(x), \\ \delta g_{\alpha\beta}(x; \epsilon^1, \dots, \epsilon^r) &= : \delta g_{\alpha\beta}(x) = -\frac{\partial \delta x^\mu}{\partial x^\alpha} g_{\mu\beta}(x) - \frac{\partial \delta x^\nu}{\partial x^\beta} g_{\alpha\nu}(x) \end{aligned} \quad (64)$$

and the continuity equation (37) which is re-written for simplicity as

$$\frac{\partial}{\partial x^\kappa} [\sqrt{-|g(x)|} J_x^\kappa] = 0 \quad (65)$$

Switching to coordinate system  $(y^0, y^1, y^2, y^3)$ , which corresponding to chart  $(V, \psi)$ , we have

$$(y^0, y^1, y^2, y^3) \xrightarrow{\pi^\lambda \circ \psi \circ \Delta \circ \psi^{-1}} \tilde{y}^\lambda = y^\lambda + \delta y^\lambda(y, \epsilon^1, \dots, \epsilon^r) =: y^\lambda + \delta y, \quad (63')$$

$$\begin{aligned} \delta u_\xi^\theta(y; \epsilon^1, \dots, \epsilon^r) &= : \delta u_\xi^\theta(y) = \frac{\partial \delta y^\theta}{\partial y^\varphi} u_\xi^\varphi(y) - \frac{\partial \delta y^\eta}{\partial y^\xi} u_\eta^\theta(y), \\ \delta g_{\alpha\beta}(y; \epsilon^1, \dots, \epsilon^r) &= : \delta g_{\alpha\beta}(y) = -\frac{\partial \delta y^\mu}{\partial y^\alpha} g_{\mu\beta}(y) - \frac{\partial \delta y^\nu}{\partial y^\beta} g_{\alpha\nu}(y) \end{aligned} \quad (64')$$

$$\frac{\partial}{\partial y^\kappa} \left[ \sqrt{-|g(y)|} J_y^\kappa \right] = 0 \quad (65')$$

For the same  $r$ -parameter family of infinitesimal diffeomorphisms of space-time manifold  $\mathbb{M}$  onto itself  $\{\Delta(\epsilon^1, \dots, \epsilon^r) =: \Delta : \mathbb{M} \rightarrow \mathbb{M} \mid |\epsilon^i| \ll 1, \forall 1 \leq i \leq r\}$ , we have

$$\delta y^\lambda = \frac{\partial(\pi^\lambda \circ \psi \circ \varphi)}{\partial x^\kappa} \delta x^\kappa = \frac{\partial y^\lambda}{\partial x^\kappa} \delta x^\kappa \quad (66)$$

From this relation it is easy to see that  $\delta x^\lambda(x; \epsilon^1, \dots, \epsilon^r)$  and  $\delta y^\lambda(y; \epsilon^1, \dots, \epsilon^r)$ , generally speaking, are functions of different forms. However, we will show that the continuity eqns. (65) and (65') are equivalent to each other.  $J_x^\kappa$  and  $J_y^\kappa$  are respectively the components in coordinate systems  $(x^0, x^1, x^2, x^3)$  and  $(y^0, y^1, y^2, y^3)$  of the same vector field.

#### Theorem 4

$$\frac{\partial}{\partial x^\kappa} [\sqrt{-|g(x)|} J_x^\kappa] = 0 \iff \frac{\partial}{\partial y^\kappa} [\sqrt{-|g(y)|} J_y^\kappa] = 0 \quad (67)$$

**Proof.** The lhs of eqn.(65) can be written as

$$\begin{aligned} \frac{\partial}{\partial x^\kappa} [\sqrt{-|g(x)|} J_x^\kappa] &= \frac{\partial y^\lambda}{\partial x^\kappa} \frac{\partial}{\partial y^\lambda} \left[ \sqrt{-|g(y)|} \left\| \frac{\partial y}{\partial x} \right\| J_x^\kappa \right] \\ &= \frac{\partial y^\lambda}{\partial x^\kappa} \frac{\partial}{\partial y^\lambda} \left[ \sqrt{-|g(y)|} J_x^\kappa \right] \left\| \frac{\partial y}{\partial x} \right\| + \frac{\partial}{\partial x^\kappa} \left\| \frac{\partial y}{\partial x} \right\| \left[ \sqrt{-|g(y)|} J_x^\kappa \right] \\ &= \frac{\partial}{\partial y^\lambda} \left[ \sqrt{-|g(y)|} \frac{\partial y^\lambda}{\partial x^\kappa} J_x^\kappa \right] \left\| \frac{\partial y}{\partial x} \right\| - \left( \frac{\partial}{\partial y^\lambda} \frac{\partial y^\lambda}{\partial x^\kappa} \right) \left[ \sqrt{-|g(y)|} J_x^\kappa \right] \left\| \frac{\partial y}{\partial x} \right\| \\ &\quad + \frac{\partial^2 y^\lambda}{\partial x^\kappa \partial x^\alpha} \frac{\partial x^\alpha}{\partial y^\lambda} \left\| \frac{\partial y}{\partial x} \right\| \left[ \sqrt{-|g(y)|} J_x^\kappa \right] \\ &= \frac{\partial}{\partial y^\lambda} \left[ \sqrt{-|g(y)|} \frac{\partial y^\lambda}{\partial x^\kappa} J_x^\kappa \right] \left\| \frac{\partial y}{\partial x} \right\| \end{aligned} \quad (68)$$

here an identity on the Jacobian

$$\frac{\partial}{\partial x^\kappa} \left| \frac{\partial y}{\partial x} \right| = \frac{\partial^2 y^\lambda}{\partial x^\kappa \partial x^\alpha} \frac{\partial x^\alpha}{\partial y^\lambda} \left| \frac{\partial y}{\partial x} \right| = \left( \frac{\partial}{\partial y^\lambda} \frac{\partial y^\lambda}{\partial x^\kappa} \right) \left| \frac{\partial y}{\partial x} \right| \quad (69)$$

has been used. Hence the proof is reduced to proving

$$\frac{\partial y^\lambda}{\partial x^\kappa} J_x^\kappa = J_y^\lambda \quad (70)$$

In fact, it's easy to show that  $\delta x^\kappa$  is a vector,  $\bar{\delta} u_\xi^\theta(x) = \tilde{u}_\xi^\theta(x) - u_\xi^\theta(x) = \delta u_\xi^\theta(x) - \delta x^\rho \partial_\rho u_\xi^\theta(x)$  is a (1,1)-tensor, and  $\bar{\delta} g_{\alpha\beta}(x) = \tilde{g}_{\alpha\beta}(x) - g_{\alpha\beta}(x) = \delta g_{\alpha\beta}(x) - \delta x^\rho \partial_\rho g_{\alpha\beta}(x)$  is a (0,2)-tensor. Hence the terms in the first brace of  $J_x^\kappa$  eqn.(38) are vectors, and the first term in the second brace is a vector too. The rest of the terms in the second brace are not vectors individually. However, their sum is a vector. This can be proven straightforwardly, though tediously. (See Appendix B) ■

**Conclusion 5** *Now we have proven:*

(i) *It is not the Noether's conservation laws written in different coordinate systems corresponding to the infinitesimal coordinate transformations with the same form that are equivalent to one another; rather, it is the Noether's conservation laws corresponding to the same family of infinitesimal diffeomorphisms of spacetime onto itself that are equivalent to one another.*

(ii) *All the Noether's conservative currents are vector fields (eqn.(70)), which should be the density, and current density of some scalar. Therefore,*

(iii) *All the Noether's conserved quantities are scalars.*

It is worth noting that the form of function  $J^\kappa$  eqn.(38) as a function of  $u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x), \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)$  is independent of coordinate systems. However, for a given family of infinitesimal diffeomorphisms of spacetime, the function forms of  $\delta x(x, \epsilon^1, \dots, \epsilon^r), \bar{\delta} u(x, \epsilon^1, \dots, \epsilon^r), \bar{\delta} g(x, \epsilon^1, \dots, \epsilon^r), \partial \bar{\delta} g(x, \epsilon^1, \dots, \epsilon^r)$  and  $\delta y(y, \epsilon^1, \dots, \epsilon^r), \bar{\delta} u(y, \epsilon^1, \dots, \epsilon^r),$

$\bar{\delta} g(y, \epsilon^1, \dots, \epsilon^r), \partial \bar{\delta} g(y, \epsilon^1, \dots, \epsilon^r)$  are generally different, hence the function forms of  $J_x^\kappa$  and  $J_y^\kappa$  as functions of field quantities, their derivatives, and coordinates, are different. This does not ruin the covariance of our presentation which is assured by the coordinate independence of eqn.(38).

We have proven these for the case of GR.. However it is easy to show they are true for the cases of SR and pre-relativity dynamics.

According to Einstein, “[w]hat we call physics comprises that group of natural sciences which base their concepts on measurements; and whose concepts and propositions lend themselves to mathematical formulation.” The aim of all natural sciences is to search for the **objective** laws of nature. In order to measure physical quantities or to describe physical processes in terms of mathematics, one needs first to choose a reference coordinate system which depends on the observer’s **subjective** will. Therefore, the ways of performing measurements and the formulations of physical laws, expressed in all reference coordinate systems should take the same form. This idea has guided Einstein from pre-relativity physics in Newtonian absolute space-time, to SR in Minkowski spacetime, and finally to GR in curved space-time. In particular, to get the proper motion equation one has to use an invariant Lagrangian  $\mathcal{L}(g(x), \partial g(x), \partial^2 g(x), u(x), \partial u(x))$ , such that the action  $A[g, u] = \int_{x(\Omega)} d^4x \sqrt{-|g(x)|} \mathcal{L}(g(x), \partial g(x), \partial^2 g(x), u(x), \partial u(x))$  is an invariant under general transformation of coordinates. The least action principle would lead to a motion equation independent of coordinate systems. The general covariance of a physical theory means the operations of measurement, and the mathematical expressions of physical laws should be independent of reference coordinate systems. It does not mean that functions of coordinates, field quantities, and their derivatives, in different coordinate systems but with the same function form, must represent the same physical geometrical object, as is generally accepted. This is not always true as has been shown above. So, Noether’s conserved current (38)

$J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)] =: J_x^\kappa$  and  $J^\kappa [u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y); \delta y, \bar{\delta} u(y), \bar{\delta} g(y), \partial \bar{\delta} g(y)] =: J_y^\kappa$  are components of the same vector field on spacetime; while the canonical energy-momentum (41)  $\tau_\rho^\kappa(x) = \tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$  and  $\tau_\rho^\kappa(y) = \tau_\rho^\kappa(u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y))$  are not components of the same tensor field on spacetime. The long standing pseudotensor problems like (61), are only a result of misreading Noether’s theorem; and the non-locality difficulty of gravitational energy-momentum does not really exist at all, had we read Noether’s theorem properly. They are results of mistaking different physical geometrical objects as one and the same.

It is important to distinguish a general law of nature and a concrete physical object or process. The former is independent of coordinates, while the latter might look different to different observers (from different coordinate systems). Yet we still can observe and describe a concrete instance in a way



independent of coordinate systems.

Now we are in a position to address the problems of conservation of energy-momentum and the gravitational energy-momentum in GR. These will be done in a later paper.

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## A Appendix

The Lagrangian density of classical fields  $L$ , is a function of the coordinates, field quantities, and their derivatives of up to the  $N$ -th order. However, because not all the variables are independent, such as  $\partial_\mu \partial_\nu u_\eta^\xi(x) = \partial_\nu \partial_\mu u_\eta^\xi(x)$ ,  $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ , etc., there are infinitely many different function forms for  $L$ . This causes indefiniteness of derivatives, such as  $\frac{\partial}{\partial \partial_\mu \partial_\nu u_\eta^\xi(x)} L$ ,  $\frac{\partial}{\partial g_{\alpha\beta}(x)} L$ . If we drop all the redundant variables, then the Einstein summation convention can no longer be used, and the expressions will become awfully complicated, especially for a large  $N$ . In order to keep the formulae neat, physicists usually treat it in a different way. Here we will illustrate their method by using the lagrangian density for vacuum Einstein's equation,  $R$  (Ricci's scalar curvature).

$R$  is a function of 16  $g_{\alpha\beta}(x)$ 's, 64  $\partial_\mu g_{\alpha\beta}(x)$ 's, and 256  $\partial_\mu \partial_\nu g_{\alpha\beta}(x)$ 's. Because  $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ , and  $\partial_\mu \partial_\nu g_{\alpha\beta}(x) = \partial_\nu \partial_\mu g_{\alpha\beta}(x)$ , there are only 150 independent variables among them. We will choose 10  $g_{\alpha\beta}(x)$ 's, 40  $\partial_\mu g_{\alpha\beta}(x)$ 's,

and 100  $\partial_\mu \partial_\nu g_{\alpha\beta}(x)$ 's ( $\alpha \leq \beta, \mu \leq \nu$ ), for the independent variables. As a function of 336 variables (As a function defined on a 336-dimensional domain),  $R$  can take infinite different forms, say,  $\varphi, \psi, \dots$ . When restricted to the 150-dimensional “sub-domain”  $D$ , all of them are the same function of 150 variables.

$$R|_D = \varphi|_D = \psi|_D = \dots \quad (\text{A1})$$

Substituting the 150 independent variables for all the variables in  $\varphi, \psi, \dots$ , we get a unique function

$$\underline{R}(g_{\alpha\beta}(x), \partial_\mu g_{\alpha\beta}(x), \partial_\mu \partial_\nu g_{\alpha\beta}(x)), \quad (\alpha \leq \beta, \mu \leq \nu) \quad (\text{A2})$$

Substituting  $\frac{1}{2}(g_{\alpha\beta}(x) + g_{\beta\alpha}(x))$ ,  $\frac{1}{2}(\partial_\mu g_{\alpha\beta}(x) + \partial_\mu g_{\beta\alpha}(x))$ , and  $\frac{1}{4}(\partial_\mu \partial_\nu g_{\alpha\beta}(x) + \partial_\mu \partial_\nu g_{\beta\alpha}(x) + \partial_\nu \partial_\mu g_{\alpha\beta}(x) + \partial_\nu \partial_\mu g_{\beta\alpha}(x))$  for  $g_{\alpha\beta}(x)$ ,  $\partial_\mu g_{\alpha\beta}(x)$ , and  $\partial_\mu \partial_\nu g_{\alpha\beta}(x)$  in  $\underline{R}$ , respectively, we get a unique function of all 336 variables, denoted by  $R(g(x), \partial g(x), \partial^2 g(x))$ . This “standard”  $R(g(x), \partial g(x), \partial^2 g(x))$  has the following property.

$$\begin{aligned} \frac{\partial R}{\partial g_{\alpha\beta}(x)} &= \frac{\partial R}{\partial g_{\beta\alpha}(x)} = \frac{1}{2} \frac{\partial \underline{R}}{\partial g_{\alpha\beta}(x)}, \alpha < \beta \\ \frac{\partial R}{\partial g_{\alpha\alpha}(x)} &= \frac{\partial \underline{R}}{\partial g_{\alpha\alpha}(x)} \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \frac{\partial R}{\partial \partial_\mu g_{\alpha\beta}(x)} &= \frac{\partial R}{\partial \partial_\mu g_{\beta\alpha}(x)} = \frac{1}{2} \frac{\partial \underline{R}}{\partial \partial_\mu g_{\alpha\beta}(x)}, \alpha < \beta \\ \frac{\partial R}{\partial \partial_\mu g_{\alpha\alpha}(x)} &= \frac{\partial \underline{R}}{\partial \partial_\mu g_{\alpha\alpha}(x)} \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} &= \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} = \frac{\partial R}{\partial \partial_\nu \partial_\mu g_{\alpha\beta}(x)} = \\ \frac{\partial R}{\partial \partial_\nu \partial_\mu g_{\beta\alpha}(x)} &= \frac{1}{4} \frac{\partial \underline{R}}{\partial \partial_\mu g_{\alpha\beta}(x)}, \alpha < \beta, \mu < \nu \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\alpha\alpha}(x)} &= \frac{\partial R}{\partial \partial_\nu \partial_\mu g_{\alpha\alpha}(x)} = \frac{1}{2} \frac{\partial \underline{R}}{\partial \partial_\mu \partial_\nu g_{\alpha\alpha}(x)}, \mu < \nu \\ \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} &= \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} = \frac{1}{2} \frac{\partial \underline{R}}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)}, \alpha < \beta \\ \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\alpha\alpha}(x)} &= \frac{\partial \underline{R}}{\partial \partial_\mu \partial_\nu g_{\alpha\alpha}(x)} \end{aligned} \quad (\text{A6})$$

When calculating the derivatives of  $R$ , we pretend that all its 336 variables are independent. Thus the indefiniteness problem no longer exists.

From (A1), we have

$$\delta\varphi|_D = \delta\psi|_D$$

While

$$\begin{aligned}
\delta\varphi|_D &= \left[ \frac{\partial\varphi}{\partial g_{\alpha\beta}(x)} \delta g_{\alpha\beta}(x) + \frac{\partial\varphi}{\partial \partial_\mu g_{\alpha\beta}(x)} \delta \partial_\mu g_{\alpha\beta}(x) + \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} \delta \partial_\mu \partial_\nu g_{\alpha\beta}(x) \right] |_D \\
&= \left[ \sum_\alpha \frac{\partial\varphi}{\partial g_{\alpha\alpha}(x)} \delta g_{\alpha\alpha}(x) + \sum_{\alpha<\beta} \left( \frac{\partial\varphi}{\partial g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial g_{\beta\alpha}(x)} \right) \delta g_{\alpha\beta}(x) \right. \\
&\quad + \sum_\alpha \frac{\partial\varphi}{\partial \partial_\mu g_{\alpha\alpha}(x)} \delta \partial_\mu g_{\alpha\alpha}(x) + \sum_{\alpha<\beta} \left( \frac{\partial\varphi}{\partial \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\mu g_{\beta\alpha}(x)} \right) \delta \partial_\mu g_{\alpha\beta}(x) \\
&\quad + \sum_{\alpha,\mu} \frac{\partial\varphi}{\partial \partial_\mu g_{\beta\alpha}(x)} \delta \partial_\mu g_{\beta\alpha}(x) + \sum_{\alpha,\mu} \frac{\partial\varphi}{\partial \partial_\mu \partial_\mu g_{\alpha\alpha}(x)} \delta \partial_\mu \partial_\mu g_{\alpha\alpha}(x) \\
&\quad + \sum_{\alpha<\beta,\mu} \left( \frac{\partial\varphi}{\partial \partial_\mu \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\mu \partial_\mu g_{\beta\alpha}(x)} \right) \delta \partial_\mu \partial_\mu g_{\alpha\beta}(x) \\
&\quad + \sum_{\alpha,\mu<\nu} \left( \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\alpha\alpha}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\alpha\alpha}(x)} \right) \delta \partial_\mu \partial_\nu g_{\alpha\alpha}(x) \\
&\quad + \sum_{\alpha<\beta,\mu<\nu} \left( \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\alpha\beta}(x)} + \right. \\
&\quad \left. \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\beta\alpha}(x)} \right) \delta \partial_\mu \partial_\nu g_{\alpha\beta}(x) \Big] |_D \tag{A7}
\end{aligned}$$

Because all the variations on the RHS of (A7) are independent, we get

$$\begin{aligned}
\left( \frac{\partial\varphi}{\partial g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial g_{\beta\alpha}(x)} \right) |_D &= \left( \frac{\partial\psi}{\partial g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial g_{\beta\alpha}(x)} \right) |_D \\
\left( \frac{\partial\varphi}{\partial \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\mu g_{\beta\alpha}(x)} \right) |_D &= \left( \frac{\partial\psi}{\partial \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial \partial_\mu g_{\beta\alpha}(x)} \right) |_D \\
\left( \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\beta\alpha}(x)} \right) |_D &= \left( \frac{\partial\psi}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial \partial_\nu \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} + \frac{\partial\psi}{\partial \partial_\nu \partial_\mu g_{\beta\alpha}(x)} \right) |_D
\end{aligned}$$

$$= (\frac{\partial\psi}{\partial\partial_\mu\partial_\nu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial\partial_\nu\partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial\partial_\mu\partial_\nu g_{\beta\alpha}(x)} + \frac{\partial\psi}{\partial\partial_\nu\partial_\mu g_{\beta\alpha}(x)})|_D \quad (\text{A8})$$

This tells us, say,

$$\begin{aligned} (\frac{\partial R}{\partial g_{\alpha\beta}(x)}\delta g_{\alpha\beta}(x))|_D &= (\frac{\partial\varphi}{\partial g_{\alpha\beta}(x)}\delta g_{\alpha\beta}(x))|_D = (\frac{\partial\psi}{\partial g_{\alpha\beta}(x)}\delta g_{\alpha\beta}(x))|_D \\ (\frac{\partial R}{\partial\partial_\mu g_{\alpha\beta}(x)}\delta\partial_\mu g_{\alpha\beta}(x))|_D &= (\frac{\partial\varphi}{\partial\partial_\mu g_{\alpha\beta}(x)}\delta\partial_\mu g_{\alpha\beta}(x))|_D \\ &= (\frac{\partial\psi}{\partial\partial_\mu g_{\alpha\beta}(x)}\delta\partial_\mu g_{\alpha\beta}(x))|_D \\ (\frac{\partial R}{\partial\partial_\mu\partial_\nu g_{\alpha\beta}(x)}\delta\partial_\mu\partial_\nu g_{\alpha\beta}(x))|_D &= (\frac{\partial\varphi}{\partial\partial_\mu\partial_\nu g_{\alpha\beta}(x)}\delta\partial_\mu\partial_\nu g_{\alpha\beta}(x)) \\ &= (\frac{\partial\psi}{\partial\partial_\mu\partial_\nu g_{\alpha\beta}(x)}\delta\partial_\mu\partial_\nu g_{\alpha\beta}(x))|_D \quad (\text{A9}) \end{aligned}$$

and even more, such as

$$\begin{aligned} &(\frac{\partial R}{\partial\partial_\kappa g_{\alpha\beta}(x)})(\delta g_{\alpha\beta}(x) - \delta x^\rho\partial_\rho g_{\alpha\beta}(x))|_D \\ &= (\frac{\partial\varphi}{\partial\partial_\kappa g_{\alpha\beta}(x)})(\delta g_{\alpha\beta}(x) - \delta x^\rho\partial_\rho g_{\alpha\beta}(x))|_D \end{aligned}$$

where  $R(g(x), \partial g(x), \partial^2 g(x))$  is the “standard” expression for  $R$ , and  $\varphi(g(x), \partial g(x), \partial^2 g(x))$  is any expression from (A1).

## B Appendix

**Proposition 7** *Let*

$$\begin{aligned} I_x^\kappa =: & (\frac{\partial R}{\partial\partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}(x)}) \times \\ & (\delta g_{\alpha\beta}(x) - \delta x^\rho\partial_\rho g_{\alpha\beta}(x)) + \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}(x)} \partial_\mu (\delta g_{\alpha\beta}(x) - \delta x^\rho\partial_\rho g_{\alpha\beta}(x)). \quad (\text{B1}) \end{aligned}$$

*Then*

$$I_y^\lambda = \frac{\partial y^\lambda}{\partial x^\kappa} I_x^\kappa. \quad (\text{B2})$$

**Proof.**

$$R = g^{\alpha\beta}g^{\rho\sigma}(\partial_\alpha\partial_\rho g_{\beta\sigma} - \partial_\alpha\partial_\beta g_{\rho\sigma}) + g^{\alpha\beta}g^{\rho\sigma}g^{\xi\eta}(\partial_\alpha g_{\beta\rho}\partial_\sigma g_{\xi\eta} + \frac{3}{4}\partial_\alpha g_{\rho\xi}\partial_\beta g_{\sigma\eta} - \frac{1}{4}\partial_\xi g_{\alpha\beta}\partial_\eta g_{\rho\sigma} - \frac{1}{2}\partial_\rho g_{\alpha\xi}\partial_\eta g_{\beta\sigma} - \partial_\alpha g_{\beta\rho}\partial_\xi g_{\sigma\eta}) \quad (B3)$$

$$\frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}} = \frac{1}{2}(g^{\alpha\kappa}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\kappa}) - g^{\alpha\beta}g^{\kappa\mu} \quad (B4)$$

Note that  $\frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}}$  is a  $(4, 0)$ -tensor, symmetrical for  $(\kappa, \mu)$ , and for  $(\alpha, \beta)$ .

$$\begin{aligned} \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}} &= \partial_\mu g_{\xi\eta}[g^{\alpha\kappa}g^{\beta\mu}g^{\xi\eta} + g^{\alpha\beta}g^{\kappa\xi}g^{\mu\eta} + \frac{3}{2}g^{\alpha\xi}g^{\beta\eta}g^{\kappa\mu} \\ &\quad - \frac{1}{2}g^{\alpha\beta}g^{\kappa\mu}g^{\xi\eta} - g^{\alpha\xi}g^{\beta\mu}g^{\kappa\eta} - g^{\alpha\kappa}g^{\beta\xi}g^{\mu\eta} - g^{\alpha\xi}g^{\beta\kappa}g^{\mu\eta}] \\ &=: \partial_\mu g_{\xi\eta}B^{\kappa\alpha\beta\mu\xi\eta} \end{aligned} \quad (B5)$$

$$\begin{aligned} \partial_\mu \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}} &= \frac{1}{2}(\partial_\mu g^{\alpha\kappa}g^{\beta\mu} + \partial_\mu g^{\alpha\mu}g^{\beta\kappa}) - \partial_\mu g^{\alpha\beta}g^{\kappa\mu} \\ &\quad + \frac{1}{2}(g^{\alpha\kappa}\partial_\mu g^{\beta\mu} + g^{\alpha\mu}\partial_\mu g^{\beta\kappa}) - g^{\alpha\beta}\partial_\mu g^{\kappa\mu} \\ &= \partial_\mu g_{\xi\eta}[-\frac{1}{2}g^{\alpha\xi}g^{\eta\kappa}g^{\beta\mu} - \frac{1}{2}g^{\alpha\xi}g^{\eta\mu}g^{\beta\kappa} + g^{\alpha\xi}g^{\eta\beta}g^{\kappa\mu}] \\ &\quad + \partial_\mu g_{\xi\eta}[-\frac{1}{2}g^{\alpha\kappa}g^{\beta\xi}g^{\eta\mu} - \frac{1}{2}g^{\alpha\mu}g^{\beta\xi}g^{\kappa\mu} + g^{\alpha\beta}g^{\kappa\xi}g^{\eta\mu}] \\ &=: \partial_\mu g_{\xi\eta}A^{\kappa\alpha\beta\mu\xi\eta} \end{aligned} \quad (B6)$$

Note that  $A^{\kappa\alpha\beta\mu\xi\eta}$  and  $B^{\kappa\alpha\beta\mu\xi\eta}$  are  $(6, 0)$ -tensors. Let

$$\begin{aligned} C^{\kappa\alpha\beta\mu\xi\eta}(x) &=: B^{\kappa\alpha\beta\mu\xi\eta}(x) - A^{\kappa\alpha\beta\mu\xi\eta}(x) \\ &= -\frac{1}{2}g^{\alpha\xi}(x)g^{\beta\mu}(x)g^{\kappa\eta}(x) + \frac{1}{2}g^{\alpha\mu}(x)g^{\beta\xi}(x)g^{\kappa\eta}(x) \\ &\quad - \frac{1}{2}g^{\alpha\xi}(x)g^{\beta\kappa}(x)g^{\mu\eta}(x) - \frac{1}{2}g^{\alpha\kappa}(x)g^{\beta\xi}(x)g^{\mu\eta}(x) \\ &\quad + \frac{1}{2}g^{\alpha\xi}(x)g^{\beta\eta}(x)g^{\kappa\mu}(x) + g^{\alpha\kappa}(x)g^{\beta\mu}(x)g^{\xi\eta}(x) - \frac{1}{2}g^{\alpha\beta}(x)g^{\kappa\mu}(x)g^{\xi\eta}(x) \end{aligned} \quad (B7)$$

$$\Gamma_{\alpha\nu}^\nu(x) = \frac{\partial y^\beta}{\partial x^\alpha} \Gamma_{\beta\gamma}^\gamma(y) + \frac{\partial}{\partial y^\sigma} \left( \frac{\partial y^\sigma}{\partial x^\alpha} \right) \quad (\text{B8})$$

Then

$$\begin{aligned} I_x^\kappa &= [C^{\kappa\alpha\beta\mu\xi\eta}(x) \partial_\mu g_{\xi\eta}(x) - \Gamma_{\mu\nu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}}] \bar{\delta} g_{\alpha\beta}(x) \\ &\quad + \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu \bar{\delta} g_{\alpha\beta}(x) \\ &= \frac{\partial x^\kappa}{\partial y^{\kappa'}} [C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial x^\xi}{\partial y^{\xi'}} \frac{\partial x^\eta}{\partial y^{\eta'}} \frac{\partial}{\partial y^{\mu'}} \left( \frac{\partial y^{\xi''}}{\partial x^\xi} \frac{\partial y^{\eta''}}{\partial x^\eta} g_{\xi''\eta''}(y) \right) \\ &\quad - \Gamma_{\nu'\mu'}^{\nu'}(y) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)}] \bar{\delta} g_{\alpha'\beta'}(y) \\ &\quad + \frac{\partial x^\kappa}{\partial y^{\kappa'}} \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)} \frac{\partial x^\alpha}{\partial y^{\alpha'}} \frac{\partial x^\beta}{\partial y^{\beta'}} \frac{\partial}{\partial y^{\mu'}} \left[ \frac{\partial x^\alpha}{\partial y^{\alpha''}} \frac{\partial x^\beta}{\partial y^{\beta''}} \bar{\delta} g_{\alpha''\beta''}(y) \right] \\ &= \frac{\partial x^\kappa}{\partial y^{\kappa'}} \{ [C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \partial_{\mu'} g_{\xi'\eta'}(y) - \Gamma_{\nu'\mu'}^{\nu'}(y) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)}] \times \\ &\quad \bar{\delta} g_{\alpha'\beta'}(y) + \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)} \partial_{\mu'} \bar{\delta} g_{\alpha'\beta'}(y) \} + \frac{\partial x^\kappa}{\partial y^{\kappa'}} rest \\ &= \frac{\partial x^\kappa}{\partial y^{\kappa'}} I_y^{\kappa'} + \frac{\partial x^\kappa}{\partial y^{\kappa'}} rest \end{aligned}$$

where

$$\begin{aligned} rest &=: [-C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial^2 x^\xi}{\partial y^{\mu'} \partial y^{\xi'}} \frac{\partial y^{\xi''}}{\partial x^\xi} g_{\xi''\eta''}(y) \\ &\quad - C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\eta''}}{\partial x^\eta} g_{\xi''\eta''}(y) \\ &\quad + \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)}] \bar{\delta} g_{\alpha'\beta'}(y) \end{aligned} \quad (\text{B9})$$

We are going to show that *rest* vanishes. Its first term is

$$\begin{aligned} &-C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial^2 x^\xi}{\partial y^{\mu'} \partial y^{\xi'}} \frac{\partial y^{\xi''}}{\partial x^\xi} g_{\xi''\eta''}(y) \bar{\delta} g_{\alpha'\beta'}(y) \\ &= [g^{\alpha'\xi'}(y) g^{\beta'\kappa'}(y) g^{\mu'\eta'}(y) - \frac{1}{2} g^{\alpha'\xi'}(y) g^{\beta'\eta'}(y) g^{\kappa'\mu'}(y) \end{aligned}$$

$$-g^{\alpha'\kappa'}(y)g^{\beta'\mu'}(y)g^{\xi'\eta'}(y) + \frac{1}{2}g^{\alpha'\beta'}(y)g^{\kappa'\mu'}(y)g^{\xi'\eta'}(y)] \times \\ \frac{\partial^2 x^\xi}{\partial y^{\mu'} \partial y^{\xi'}} \frac{\partial y^{\xi''}}{\partial x^\xi} g_{\xi''\eta'}(y) \bar{\delta} g_{\alpha'\beta'}(y)$$

$$= g^{\alpha'\xi'}(y)g^{\beta'\kappa'}(y) \frac{\partial}{\partial x^\xi} \left( \frac{\partial x^\xi}{\partial y^{\xi'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{A})$$

$$- \frac{1}{2} g^{\alpha'\xi'}(y)g^{\kappa'\mu'}(y) \frac{\partial^2 x^\xi}{\partial y^{\mu'} \partial y^{\xi'}} \frac{\partial y^{\beta'}}{\partial x^\xi} \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{B})$$

$$-g^{\alpha'\kappa'}(y)g^{\beta'\mu'}(y) \frac{\partial}{\partial x^\xi} \left( \frac{\partial x^\xi}{\partial y^{\mu'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{A})$$

$$+ \frac{1}{2} g^{\alpha'\beta'}(y)g^{\kappa'\mu'}(y) \frac{\partial}{\partial x^\xi} \left( \frac{\partial x^\xi}{\partial y^{\mu'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad ((\text{C})) \quad (\text{B10})$$

The second term is

$$-C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\eta''}}{\partial x^\eta} g_{\xi'\eta''}(y) \bar{\delta} g_{\alpha'\beta'}(y) \\ = [g^{\alpha'\xi'}(y)g^{\beta'\kappa'}(y)g^{\mu'\eta'}(y) - \frac{1}{2}g^{\alpha'\xi'}(y)g^{\beta'\eta'}(y)g^{\kappa'\mu'}(y)$$

$$-g^{\alpha'\kappa'}(y)g^{\beta'\mu'}(y)g^{\xi'\eta'}(y) + \frac{1}{2}g^{\alpha'\beta'}(y)g^{\kappa'\mu'}(y)g^{\xi'\eta'}(y)] \times \\ \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\eta''}}{\partial x^\eta} g_{\xi'\eta''}(y) \bar{\delta} g_{\alpha'\beta'}(y)$$

$$= g^{\beta'\kappa'}(y)g^{\mu'\eta'}(y) \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\alpha'}}{\partial x^\eta} \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{D})$$

$$- \frac{1}{2} g^{\beta'\eta'}(y)g^{\kappa'\mu'}(y) \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\alpha'}}{\partial x^\eta} \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{B})$$

$$-g^{\alpha'\kappa'}(y)g^{\beta'\mu'}(y) \frac{\partial}{\partial x^\eta} \left( \frac{\partial x^\eta}{\partial y^{\mu'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{A})$$

$$+ \frac{1}{2} g^{\alpha'\beta'}(y)g^{\kappa'\mu'}(y) \frac{\partial}{\partial x^\eta} \left( \frac{\partial x^\eta}{\partial y^{\mu'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{C}) \quad (\text{B11})$$

The third term is

$$\begin{aligned}
& \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha' \beta'}(y)} \bar{\delta} g_{\alpha' \beta'}(y) \\
&= \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) (g^{\alpha' \kappa'} g^{\beta' \mu'} - g^{\alpha' \beta'} g^{\kappa' \mu'}) \bar{\delta} g_{\alpha' \beta'}(y) \\
&= \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) g^{\alpha' \kappa'} g^{\beta' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{A}) \\
&\quad - \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) g^{\alpha' \beta'} g^{\kappa' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{C}) \quad (\text{B12})
\end{aligned}$$

The forth term is

$$\begin{aligned}
& - \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'' \beta'}(y)} \frac{\partial^2 x^\alpha}{\partial y^{\mu'} \partial y^{\alpha''}} \frac{\partial y^{\alpha'}}{\partial x^\alpha} \bar{\delta} g_{\alpha' \beta'}(y) \\
&= - \frac{\partial^2 x^\alpha}{\partial y^{\mu'} \partial y^{\alpha''}} \frac{\partial y^{\alpha'}}{\partial x^\alpha} (g^{\alpha'' \kappa'} g^{\beta' \mu'} - g^{\alpha'' \beta'} g^{\kappa' \mu'}) \bar{\delta} g_{\alpha' \beta'}(y) \\
&= - \frac{\partial^2 x^\alpha}{\partial y^{\mu'} \partial y^{\alpha''}} \frac{\partial y^{\alpha'}}{\partial x^\alpha} g^{\alpha'' \kappa'} g^{\beta' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{B}) \\
&\quad + \frac{\partial^2 x^\alpha}{\partial y^{\mu'} \partial y^{\alpha''}} \frac{\partial y^{\alpha'}}{\partial x^\alpha} g^{\alpha'' \beta'} g^{\kappa' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{B}) \quad (\text{B13})
\end{aligned}$$

The last term is

$$\begin{aligned}
& - \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha' \beta''}(y)} \frac{\partial^2 x^\beta}{\partial y^{\mu'} \partial y^{\beta''}} \frac{\partial y^{\beta'}}{\partial x^\beta} \bar{\delta} g_{\alpha' \beta'}(y) \\
&= - \frac{\partial^2 x^\beta}{\partial y^{\mu'} \partial y^{\beta''}} \frac{\partial y^{\beta'}}{\partial x^\beta} (g^{\alpha' \kappa'} g^{\beta'' \mu'} - g^{\alpha' \beta''} g^{\kappa' \mu'}) \bar{\delta} g_{\alpha' \beta'}(y) \\
&= - \frac{\partial^2 x^\beta}{\partial y^{\mu'} \partial y^{\beta''}} \frac{\partial y^{\beta'}}{\partial x^\beta} g^{\alpha' \kappa'} g^{\beta'' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{D}) \\
&\quad + \frac{\partial^2 x^\beta}{\partial y^{\mu'} \partial y^{\beta''}} \frac{\partial y^{\beta'}}{\partial x^\beta} g^{\alpha' \beta''} g^{\kappa' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{B}) \quad (\text{B14})
\end{aligned}$$

All the terms marked (A) cancel each other, all the terms marked (B) cancel each other, etc. Therefore we get

$$I_x^\kappa = \frac{\partial x^\kappa}{\partial y^{\kappa'}} I_y^{\kappa'} \quad (\text{B15})$$

That is (B2). ■



# Pseudotensor Problem of Gravitational Energy-momentum and Noether's Theorem Revisited

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## Abstract

Based on a general variational principle, Noether's theorem is revisited. It is shown that the so called pseudotensor problem of the gravitational energy-momentum is a result of mis-reading Noether's theorem; and in fact, all the Noether's conserved quantities in general relativity are scalars. It is also shown that the non-localizability of gravitational energy-momentum can not be attributed to the equivalence principle, by using a counter-example. As a direct consequence of variational principle, a generalized Hamilton-Jacobi equation for the Hamilton's principal functional is obtained.

## 1 Introduction

Ever since Einstein introduced the gravitational energy-momentum complex to keep the law of conservation of energy-momentum alive in general relativity (GR) in 1918, the so-called pseudotensor problem of various gravitational energy-momentum complexes has been troubling relativists [1]. One of the consequences of pseudotensor property is the non-localizability of gravitational energy-momentum. It is attributed to the equivalence principle. Some

relativitists think it is inherent in the theory of GR. However, not all relativitists accept this viewpoint. It is widely accepted that the proof of the positivity of the total gravitational energy both in spacial and null infinity is one of the greatest achievements in classical GR in the last 30 years. This success inspired efforts to find quasi-local gravitational energy-momentum. However, finding quasi-local gravitational quantities has proven to be surprisingly difficult [2]. Experimentally, the gravitational wave carrying energy-momentum has not been tested directly[3]. The aim of this article is to explore the pseudotensor problem of the gravitational energy-momentum, in particular, the difficulty of non-localizability of the gravitational energy-momentum. The whole argument is based on the variational principle of dynamics. It is shown that the non-localizability of gravitational energy-momentum can not be attributed to the equivalence principle, by using a counter-example. It is also shown that the so-called pseudotensor problem of the gravitational energy-momentum is only a result of mis-reading Noether's theorem, and mistaking different geometric physical objects as one and the same; hence the non-localizability difficulty does not exist at all. In fact, all Noether's conserved quantities in GR are scalars. As a direct consequence of variational principle, a generalized Hamilton-Jacobi equation for the Hamilton's principal functional is obtained; and it will be used for the quantization of fields in GR in a later work.

The history of modern physics has proven that the variational principle approach to dynamics is not only an alternative and equivalent version to the naive, intuitive approach, but also yields deeper insights into the underlying physics. For instance, it is hard to imagine that the statistical mechanics could have been established without using the concept of phase space, and that the quantum mechanics could have been established without using the concept of Hamiltonian. Therefore, we will base our argument on a general variational principle for the classical field. It might be for the similar consideration that soon after Einstein proposed his general theory of relativity, Hilbert made the first successful attempt to get Einstein's equation by using the least action principle. The Lagrangian being used for vacuum Einstein's equation,  $(16\pi G)^{-1}R$ , is the only independent scalar constructed in terms of the metric field and its derivatives of no higher than the second order. However, because the Ricci scalar curvature  $R$  contains the second order derivatives of the metric field  $g_{\mu\nu}(x)$ , which is now the dynamic variable along with the matter field, the least action principle for Lagrangians containing only the field quantity and its first order derivatives does not lead

to Einstein's field equation. The generally accepted solution to this difficulty is adding the Gibbons-Hawking boundary term to the Hilbert action and keeping the least action principle unchanged[4]. But there is another solution to this difficulty, which was initiated by Hilbert and will be adopted in the present paper. The least action principle will be restated and the Hilbert action will still be used for the vacuum Einstein's equation. In order to show this is proper and natural, we will consider the variational principle for classical fields in  $n(\geq 2)$ -dimensional spacetime with a Lagrangian containing the field quantity and its derivatives of up to the  $N(\geq 1)$ -th order. In our opinion, non-local interaction is not acceptable, so we assume that the Lagrangian does not contain the integration of the field quantity.

There have been varied versions of variational principle and Noether's theorem in the literature[5], and different notations have been used by different authors. For the readers' convenience, and for the consistency of the reasoning, we start with presenting a general variational principle for classical fields in  $n(\geq 2)$ -dimensional spacetime with a Lagrangian containing the field quantity and its derivatives of up to the  $N(\geq 1)$ -th order. In section 2, the Lagrangian formalism is presented, and a generalized Hamilton-Jacobi equation for Hamilton's principal functional is obtained. In section 3, the Noether's theorem is rederived. Then, in section 4, these general results are applied to the specific case of general relativity, especially Noether's theorem is applied to get quite a few conservation laws in GR. It is noted that most of the conserved quantities obtained here are not tensors like various gravitational energy-momentum complexes. By using a counter-example, we show that the non-localizability of gravitational energy-momentum can not be simply attributed to the equivalence principle. Then Noether's theorem is revisited in section 5, and it is shown that the so-called pseudotensor problem of varied gravitational energy-momentum complexes is a result of taking different geometrical physical objects as one and the same; as a matter of fact, all the Noether's conserved quantities in GR are scalars; and the non-locality difficulty does not really exist.

These results will be used to explore the energy-momentum conservation and the gravitational energy-momentum in GR in a later paper.

## 2 A general variational principle for classical fields

### 2.1 Lagrangian formulation

First, we present a useful mathematical formula for the variational principle, which does not rely on physics.

Suppose  $\{\Phi_B : \mathbb{R}^n \rightarrow \mathbb{R} \text{ (or } \mathbb{C}) \mid B = 1, 2, \dots, f\}$  ( $n \geq 2$ ) are smooth functions, and function  $L = L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x))$  is smooth with respect to all its arguments. It is easy to show just by using Leibniz's rule that (the Einstein convention is used for coordinate indices)<sup>1</sup>

$$\begin{aligned}
\delta L &= \sum_{B=1}^f \sum_{X=0}^N \delta \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x) \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\
&= \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} + \partial_{\lambda} \sum_{B=1}^f \sum_{Z=0}^{N-1} \\
&\quad \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) \sum_{Y=0}^{N-1-Z} (-1)^Y \partial_{\mu_1} \cdots \partial_{\mu_Y} \frac{\partial L}{\partial \partial_{\lambda} \partial_{\mu_1} \cdots \partial_{\mu_Y} \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x)} \\
&=: \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\
&\quad + \partial_{\lambda} \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \tag{1}
\end{aligned}$$

---

<sup>1</sup>In order to avoid the indefiniteness of derivatives such as  $\frac{\partial}{\partial \partial_1 \partial_2 \Phi_B} L$ , and in order to keep the formulae neat, it is assumed in the present paper without loss of generality that

$$\begin{aligned}
&L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x)) \\
&= : L(x^\alpha, \Phi_B(x), \partial_{\alpha_1} \Phi_B(x), \partial_{\alpha_1} \partial_{\alpha_2} \Phi_B(x), \dots, \partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_N} \Phi_B(x)) \\
&= L(x^\alpha, \Phi_B(x), \partial_{\alpha} \Phi_B(x), \frac{1}{2!} \sum_{P \in S_2} \partial_{\alpha_{P(1)}} \partial_{\alpha_{P(2)}} \Phi_B(x), \dots, \\
&\quad \frac{1}{N!} \sum_{P \in S_N} \partial_{\alpha_{P(1)}} \partial_{\alpha_{P(2)}} \cdots \partial_{\alpha_{P(N)}} \Phi_B(x))
\end{aligned}$$

We will pretend that all the cross derivatives are independent variables of  $L$ , when calculating  $\frac{\partial}{\partial \partial_1 \partial_2 \Phi_B} L$ , etc. See Appendix A for the details.

when  $\Phi_B(x) \mapsto \tilde{\Phi}_B(x) = \Phi_B(x) + \delta\Phi_B(x), \forall B = 1, \dots, f$ . Consider the functional  $F$  of the following form

$$F[\Phi] = \int_{\Omega} d^n x L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x)), \quad (2)$$

where  $\Omega$  is an open subset with a compact closure of  $\mathbb{R}^n$ . When the arguments  $\Phi_B(x) \forall B = 1, \dots, f$  change slightly, the variation of functional  $F$  is

$$\begin{aligned} \delta F[\Phi] &= \int_{\Omega} d^n x \sum_{B=1}^f \sum_{X=0}^N \delta\partial_{\lambda_1} \dots \partial_{\lambda_X} \Phi_B(x) \frac{\partial L}{\partial\partial_{\lambda_1} \dots \partial_{\lambda_X} \Phi_B(x)} \\ &= \int_{\Omega} d^n x \sum_{B=1}^f \delta\Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \dots \partial_{\lambda_X} \frac{\partial L}{\partial\partial_{\lambda_1} \dots \partial_{\lambda_X} \Phi_B(x)} \\ &\quad + \int_{\Omega} d^n x \partial_{\lambda} \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta\partial_{\nu_1} \dots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \\ &= \int_{\Omega} d^n x \sum_{B=1}^f \delta\Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \dots \partial_{\lambda_X} \frac{\partial L}{\partial\partial_{\lambda_1} \dots \partial_{\lambda_X} \Phi_B(x)} \\ &\quad + \int_{\partial\Omega} ds_{\lambda}(x) \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta\partial_{\nu_1} \dots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \end{aligned} \quad (3)$$

This can be easily obtained by using eqn.(1) and the Stokes theorem. The derivatives of functional (2) is defined as follows.

**Definition 1** *If the change of the functional (2) can be expressed as*

$$F[\Phi + \delta\Phi] - F[\Phi] = \int_{\Omega} d^n x \sum_{B=1}^f \delta\Phi_B(x) D^B[\Phi, x] + o[\delta\Phi] \quad (4)$$

where  $D^B[\Phi, x]$  is a functional of  $\Phi$  varying with  $x$ , and  $o[\delta\Phi]$  is a higher order infinitesimal of  $\delta\Phi$ , when  $\{\Phi_B \mid B = 1, 2, \dots, f\}$  change slightly while the boundary values of  $\Phi, \partial\Phi, \dots, \partial^{N-1}\Phi$  are kept fixed, then  $F$  is called differentialble at  $\Phi$ , and  $D^B[\Phi, x]$  is called the derivative of functional  $F$  with respect to  $\Phi_B$  at  $\Phi$  and point  $x$ , and denoted by

$$D^B[\Phi, x] = \frac{\delta F[\Phi]}{\delta \Phi_B(x)} \quad (5)$$

Let us now apply the general formula (3) to the action functional of classical field  $\{\Phi_B : M \longrightarrow \mathbb{R} \text{ (or } \mathbb{C}) \mid B = 1, 2, \dots, f\}$

$$A[\Phi] = \int_{x(\Omega)} d^n x L(x, \Phi(x), \partial \Phi(x), \partial^2 \Phi(x), \dots, \partial^N \Phi(x)) \quad (6)$$

where  $\Omega$  is an open subset with a compact closure of the spacetime manifold  $M$ ,  $x(\Omega) \subset \mathbb{R}^n$  is the image of  $\Omega \subset M$  under the coordinate mapping  $x : M \rightarrow \mathbb{R}^n$  and  $L$  is the Lagrangian of the field. We get the difference between the action functionals over  $\Omega$  of two conceivable movements (two paths allowed by the constraints) close to each other

$$\begin{aligned} \delta A[\Phi] &= \int_{x(\Omega)} d^n x \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\ &\quad + \int_{x(\Omega)} d^n x \partial_\lambda \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \\ &= \int_{x(\Omega)} d^n x \sum_{B=1}^f \delta \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} \\ &\quad + \int_{x(\partial\Omega)} ds_\lambda(x) \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \end{aligned} \quad (7)$$

Equation (7) suggests that for all  $N \geq 1$ ,  $n \geq 2$  **the least action principle** read as follows.

*For any spacetime region  $\Omega$ , among all possible movements in  $\Omega$  with the same boundary condition*

$$\delta \Phi|_{\partial\Omega} = 0, \delta \partial \Phi|_{\partial\Omega} = 0, \dots, \delta \partial^{N-1} \Phi|_{\partial\Omega} = 0, \quad (8)$$

*the real movement (the path allowed by physical laws) corresponds to the stationary value of the action over  $\Omega$ .*

Combining eqns.(7), (8), one obtains **the field equation (Euler-Lagrange equation)** satisfied by the real movements

$$\frac{\delta A[\Phi]}{\delta \Phi_B(x)} = \sum_{X=0}^N (-1)^X \partial_{\lambda_1} \cdots \partial_{\lambda_X} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_X} \Phi_B(x)} = 0. \quad (9)$$

This Lagrangian formalism applies to Newtonian dynamics, dynamics in special relativity (SR) and dynamics in general relativity. Specific covariance is the heritage from the Lagrangian being used. It is not the heritage from this general formalism.

## 2.2 Hamilton's principal functional and Hamilton-Jacobi's equation

Let us consider the difference between actions over spacetime region  $\Omega$  of two real movements close to each other. Using eqns.(7) and (9), one gets, for real movements

$$\delta A[\Phi] = \int_{x(\partial\Omega)} ds_\lambda(x) \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \quad (10)$$

From eqn.(10), one sees that the action over a spacetime region  $\Omega$  of a real movement is determined by the closed hyper-surface  $\partial\Omega$ , and  $\Phi|_{\partial\Omega}$ ,  $\partial\Phi|_{\partial\Omega}, \dots, \partial^{N-1}\Phi|_{\partial\Omega}$ . It will be called the generalized Hamilton's principal functional and denoted by

$$S = S[\partial\Omega, \Phi|_{\partial\Omega}, \partial\Phi|_{\partial\Omega}, \dots, \partial^{N-1}\Phi|_{\partial\Omega}] \quad (11)$$

Re-write eqn.(10) as

$$\delta S = \int_{x(\partial\Omega)} ds_\lambda(x) \sum_{B=1}^f \sum_{Z=0}^{N-1} \delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x) \quad (12)$$

Note that when  $\Phi_B|_{\partial\Omega}$  is given, only one of the  $n$  derivatives  $\partial_\lambda \Phi_B|_{\partial\Omega}$  ( $\lambda = 0, 1, \dots, n-1$ ) is independent; when  $\partial_\lambda \Phi_B|_{\partial\Omega}$  is given, only one of the  $n$  derivatives  $\partial_\mu \partial_\lambda \Phi_B|_{\partial\Omega}$  ( $\mu = 0, 1, \dots, n-1$ ) is independent; and so on. Thus for a given suffix  $B$ , only  $N$  items (  $\Phi_B|_{\partial\Omega}$ , one of  $\partial_{\lambda_1} \Phi_B|_{\partial\Omega}, \dots$ , one of  $\partial_{\lambda_1} \cdots \partial_{\lambda_{N-1}} \Phi_B|_{\partial\Omega}$  ( $\lambda_j = 0, 1, \dots, n-1$ ) ) are independent.

In order to formulate the generalized Hamilton-Jacobi's equation, one needs a new type of functional derivative[6].

**Definition 2** Let  $\Sigma$  be a closed hypersurface in  $\mathbb{R}^n$ ,  $\Psi$  a function defined on  $\mathbb{R}^n$ , and  $F = F[\Sigma, \Psi|_\Sigma]$  a functional of  $\Sigma$  and  $\Psi|_\Sigma$ . The functional derivatives are defined as follows. If the variation of  $F$  can be written as

$$\delta F[\Sigma, \Psi|_\Sigma] = \int_{\Sigma} ds_{\lambda}(x) \{Y[\Sigma, \Psi|_{\Sigma}, x]_{\mu}^{\lambda} \delta \Sigma^{\mu}(x) + Z[\Sigma, \Psi|_{\Sigma}, x]^{\lambda} \delta \Psi(x)\} \quad (13)$$

where  $Y[\Sigma, \Psi|_{\Sigma}, x]_{\mu}^{\lambda}$  and  $Z[\Sigma, \Psi|_{\Sigma}, x]^{\lambda}$  are functionals of  $\Sigma$  and  $\Psi|_{\Sigma}$ , which vary with  $x$ , when  $\Sigma \mapsto \tilde{\Sigma}$ , and  $\Psi \mapsto \tilde{\Psi}$ , then they are called the functional derivative of  $F$  with respect to  $\Sigma^{\mu}(x)$  and  $\Psi(x)$ , and denoted by

$$Y[\Sigma, \Psi|_{\Sigma}, x]_{\mu}^{\lambda} = \left( \frac{\delta F}{\delta \Sigma^{\mu}(x)} \right)^{\lambda}, \quad Z[\Sigma, \Psi|_{\Sigma}, x]^{\lambda} = \left( \frac{\delta F}{\delta \Psi(x)} \right)^{\lambda} \quad (14)$$

respectively.

Hence we have

$$\delta F[\Sigma, \Psi|_{\Sigma}] = \int_{\Sigma} ds_{\lambda}(x) \left[ \left( \frac{\delta F}{\delta \Sigma^{\mu}(x)} \right)^{\lambda} \delta \Sigma^{\mu}(x) + \left( \frac{\delta F}{\delta \Psi(x)} \right)^{\lambda} \delta \Psi(x) \right] \quad (15)$$

The hypersurface  $\Sigma$  is given by the parameter equation

$$x^{\mu} = \Sigma^{\mu}(\theta^1, \theta^2, \theta^3) \quad (16)$$

The  $\delta \Sigma^{\mu}(x)$  and  $\delta \Psi(x)$  in eqn.(13) are respectively

$$\begin{aligned} \delta \Sigma^{\mu}(x) &= \tilde{\Sigma}^{\mu}(\theta^1, \theta^2, \theta^3) - \Sigma^{\mu}(\theta^1, \theta^2, \theta^3), \\ \delta \Psi(x) &= \tilde{\Psi}(\tilde{\Sigma}^{\mu}(\theta^1, \theta^2, \theta^3)) - \Psi(\Sigma^{\mu}(\theta^1, \theta^2, \theta^3)). \end{aligned} \quad (17)$$

Now, from eqns.(12) and (15) we get  $[\partial \Omega$  in (12) is  $\Sigma$  in (15)]

$$\begin{aligned} \left( \frac{\delta S}{\delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x)} \right)^{\lambda} &= K^{B\lambda\nu_1 \dots \nu_Z}(x), \\ \forall Z &= 0, \dots, N-1, B = 1, \dots, f. \end{aligned} \quad (18)$$

Following the evolution of one real movement and observing the change of its action with  $\Sigma$ ,



$$\begin{aligned}
\delta S &= \int_{x(\partial\Omega)} ds_\lambda(x) L \delta_\sigma^\lambda \delta \Sigma^\sigma(x) \\
&= \int_{x(\partial\Omega)} ds_\lambda(x) \left[ \left( \frac{\delta S}{\delta \Sigma^\sigma(x)} \right)^\lambda \delta \Sigma^\sigma(x) \right. \\
&\quad \left. + \sum_{B=0}^f \sum_{Z=0}^{N-1} \left( \frac{\delta S}{\delta \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x)} \right)^\lambda \partial_\sigma \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) \delta \Sigma^\sigma(x) \right] \quad (19)
\end{aligned}$$

We get the generalized Hamilton-Jacobi's equation.

$$\left( \frac{\delta S}{\delta \Sigma^\sigma(x)} \right)^\lambda + [L \delta_\sigma^\lambda - \sum_{B=0}^f \sum_{Z=0}^{N-1} \partial_\sigma \partial_{\nu_1} \cdots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1 \cdots \nu_Z}(x)] = 0 \quad (20)$$

It will be shown in the next section that the expression in the bracket is just the canonical energy-momentum  $\tau_\sigma^\lambda(x)$ , hence we have

$$\left( \frac{\delta S}{\delta \Sigma^\sigma(x)} \right)^\lambda + \tau_\sigma^\lambda(x) = 0. \quad (21)$$

For a dynamic system with finite degrees of freedom, we have the Hamilton's principal function  $S(q^1, \dots, q^h, t)$ , and the Hamilton-Jacobi's equation is a partial differential equation  $\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0$ . While for the classical field, we have the Hamilton's principal functional, and the Hamilton-Jacobi's equation is a set of differential equations for functional. It plays an important role in canonical quantization for fields.

## 3 Noether's theorem

### 3.1 Re-deriving the theorem

There have been varied versions of Noether's theorem, and different notations have been used. To avoid confusion, here we redrive Noether's theorem for classical fields in  $n$ -dimensional spacetime with a Lagrangian containing field quantity and its derivatives of up to the  $N$ -th order, in terms of coordinate language.

**Theorem 3** *If the action of classical fields over every spacetime region  $\Omega$  with a compact closure*

$$A[\Phi] = \int_{x(\Omega)} d^n x L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x))$$

*remains unchanged under the following  $r$ -parameter family of infinitesimal transformations of coordinates and field quantities*

$$\begin{aligned} x^\lambda &\longmapsto \tilde{x}^\lambda = x^\lambda + \delta x^\lambda, \\ \delta x^\lambda &=: \delta x^\lambda(x, \epsilon^1, \dots, \epsilon^r), |\epsilon^i| \ll 1, \delta x^\lambda(x, 0, \dots, 0) = 0 \\ \Phi_B(x) &\longmapsto \tilde{\Phi}_B(\tilde{x}) = \Phi_B(x) + \delta\Phi_B(x), \\ \delta\Phi_B(x) &=: \delta\Phi_B(x, \epsilon^1, \dots, \epsilon^r), |\epsilon^i| \ll 1, \delta\Phi_B(x, 0, \dots, 0) = 0 \end{aligned} \quad (22)$$

*then there exist  $r$  conservation laws.*

**Proof.** The small change of field quantity  $\delta\Phi_B(x)$  can be divided into two parts, the part due to the small change of its function form and the part due to the small change of its arguments respectively.

$$\delta\Phi_B(x) = \bar{\delta}\Phi_B(x) + \delta x^\sigma \partial_\sigma \Phi_B(x) \quad (23)$$

Similarly the small change of derivatives of field quantity  $\delta[\partial_{\lambda_1}\partial_{\lambda_2}\cdots\partial_{\lambda_X}\Phi_B(x)]$  can be written as

$$\begin{aligned} \delta[\partial_{\lambda_1}\partial_{\lambda_2}\cdots\partial_{\lambda_X}\Phi_B(x)] &= \partial_{\lambda_1}\partial_{\lambda_2}\cdots\partial_{\lambda_X}\bar{\delta}\Phi_B(x) \\ &\quad + \delta x^\sigma \partial_\sigma \partial_{\lambda_1}\partial_{\lambda_2}\cdots\partial_{\lambda_X}\Phi_B(x) \end{aligned} \quad (24)$$

The variation of the action can be divided into two parts. One is due to the small change of the integration domain  $x(\Omega) \longmapsto \tilde{x}(\Omega)$  in  $\mathbb{R}^n$ , and the other is due to the small change of the integrand

$$\begin{aligned} &L(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x)) \\ &\longmapsto L(x, \tilde{\Phi}(x), \partial\tilde{\Phi}(x), \partial^2\tilde{\Phi}(x), \dots, \partial^N\tilde{\Phi}(x)) \end{aligned}$$

$$\begin{aligned}
\delta A[\Phi] &= \int_{x(\partial\Omega)} ds_\lambda(x) \delta x^\lambda L \\
&+ \int_{x(\Omega)} d^n x \sum_{B=1}^f \sum_{X=0}^N \frac{\partial L}{\partial \partial_{\mu_1} \cdots \partial_{\mu_X} \Phi_B(x)} \partial_{\mu_1} \cdots \partial_{\mu_X} \bar{\delta} \Phi_B(x) \\
&= \int_{x(\Omega)} d^n x \sum_{B=1}^f \bar{\delta} \Phi_B(x) \sum_{X=0}^N (-1)^X \partial_{\mu_1} \cdots \partial_{\mu_X} \frac{\partial L}{\partial \partial_{\mu_1} \cdots \partial_{\mu_X} \Phi_B(x)} \\
&+ \int_{x(\Omega)} d^n x \partial_\lambda [\delta x^\sigma \delta_\sigma^\lambda L + \sum_{B=1}^f \sum_{Z=0}^{N-1} \partial_{\nu_1} \cdots \partial_{\nu_Z} \bar{\delta} \Phi_B(x) K^{B\lambda\nu_1 \dots \nu_Z}(x)] \quad (25)
\end{aligned}$$

The first integrand at rhs vanishes for real movement, hence the second integral does too. One gets the following continuity equation due to the arbitrariness of  $\Omega$ .

$$\partial_\lambda [\delta x^\sigma \delta_\sigma^\lambda L + \sum_{B=1}^f \sum_{Z=0}^{N-1} \partial_{\nu_1} \cdots \partial_{\nu_Z} (\delta \Phi_B(x) - \delta x^\sigma \partial_\sigma \Phi_B(x)) K^{B\lambda\nu_1 \dots \nu_Z}(x)] = 0$$

or

$$\begin{aligned}
&\int_{x(\partial\Omega)} ds_\lambda(x) \left\{ \delta x^\sigma \delta_\sigma^\lambda L + \sum_{B=1}^f \sum_{Z=0}^{N-1} \right. \\
&\quad \left. [\partial_{\nu_1} \cdots \partial_{\nu_Z} (\delta \Phi_B(x) - \delta x^\sigma \partial_\sigma \Phi_B(x)) K^{B\lambda\nu_1 \dots \nu_Z}(x)] \right\} = 0 \quad (26)
\end{aligned}$$

Noting that both  $\delta x^\sigma$  and  $\delta \Phi_B(x)$  depend on  $r$  real parameters, one gets from eqn.(26)  $r$  conservation laws.

$$\begin{aligned}
&\partial_\lambda \left\{ \sum_{B=1}^f \sum_{Z=0}^{N-1} \partial_{\nu_1} \cdots \partial_{\nu_Z} \left( \frac{\partial \delta \Phi_B(x)}{\partial \epsilon^\alpha} \Big|_{\epsilon=0} - \frac{\partial \delta x^\sigma}{\partial \epsilon^\alpha} \Big|_{\epsilon=0} \partial_\sigma \Phi_B(x) \right) K^{B\lambda\nu_1 \dots \nu_Z}(x) \right. \\
&\quad \left. + \frac{\partial \delta x^\sigma}{\partial \epsilon^\alpha} \Big|_{\epsilon=0} \delta_\sigma^\lambda L \right\} = 0, \quad \forall \alpha = 1, \dots, r \quad (27)
\end{aligned}$$

■

### 3.2 Conservation law due to “coordinate shift” invariance

In this subsection, we restrict our discussion to those cases, such that the Lagrangian does not manifestly contain coordinates and the action of classical

field over any spacetime region

$$A[\Phi] = \int_{x(\Omega)} d^n x L(\Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x)) \quad (28)$$

remains unchanged under the following “coordinate shift”.

$$\delta x^\sigma = \varepsilon^\sigma, \quad \delta\Phi_B(x) = 0. \quad (29)$$

And eqn.(27) reads

$$\partial_\lambda \tau_\sigma^\lambda(x) = 0, \quad (30)$$

where

$$\tau_\sigma^\lambda(x) = \delta_\sigma^\lambda L - \sum_{B=1}^f \sum_{Z=0}^{N-1} \partial_\sigma \partial_{\nu_1} \dots \partial_{\nu_Z} \Phi_B(x) K^{B\lambda\nu_1\dots\nu_Z}(x) \quad (31)$$

is usually called canonical energy-momentum.

Comparing eqn.(20) and (31), we get eqn.(21), the generalized Hamilton-Jacobi equation.

The formalism presented so far is good for any classical field in  $n$ -dimensional spacetime with a Lagrangian containing field quantities and their derivatives of up to the  $N$ -th order, no matter it is Galileo covariant, Lorentz covariant, general covariant or without any covariance. Therefore, we can not say eqn.(21) is covariant even though it looks like so. The covariance of the formalism is the heritage from the Lagrangian being used, not the heritage from the formalism itself.

## 4 Variational principle approach to general relativity

Let us apply the general results obtained above to the classic fields in GR. We will consider the case where the matter field is a  $(1, 1)$ -tensor field. The results can be readily generalized to any  $(r, s)$ -tensor matter field. For the dynamic system,  $(1, 1)$ -tensor matter field  $u_\xi^\theta(x)$  plus the metric field  $g_{\alpha\beta}(x)$ , the Lagrangian and the action over any spacetime region  $\Omega$  are

$$\begin{aligned}
& L(g(x), \partial g(x), \partial^2 g(x), u(x), \partial u(x)) \\
&= \sqrt{-|g(x)|} [\mathcal{L}(g(x), u(x), \nabla u(x)) + \frac{1}{16\pi G} R] = L_M + L_G, \quad (32)
\end{aligned}$$

$$\begin{aligned}
A[g, u] &= \int_{x(\Omega)} d^4x \sqrt{-|g(x)|} [\mathcal{L}(g(x), u(x), \nabla u(x)) \\
&\quad + \frac{1}{16\pi G} R] = A_M[g, u] + A_G[g] \quad (33)
\end{aligned}$$

where  $R$  is the Ricci scalar curvature,  $\mathcal{L}(g(x), u(x), \nabla u(x))$  is the Lagrangian for matter field obtained from the Lagrangian in special relativity

$\mathcal{L}(\eta, u(x), \partial u(x))$  by replacing the Minkowski metric  $\eta = \text{diag}(-1, 1, 1, 1)$  with  $g_{\alpha\beta}(x)$ , and replacing the partial derivatives  $\partial_\mu u_\xi^\theta(x)$  with the covariant derivatives  $\nabla_\mu u_\xi^\theta(x)$ .

#### 4.1 Einstein's field equation

The Euler-Lagrange equation, Eqn.(9) now reads

$$\frac{\delta A[g, u]}{\delta u_\varphi^\theta(x)} = \sqrt{-|g(x)|} \left[ \frac{\partial \mathcal{L}}{\partial u_\varphi^\theta(x)} - \nabla_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\varphi^\theta(x)} \right] = 0 \quad (34)$$

$$\frac{\delta A[g, u]}{\delta g_{\alpha\beta}(x)} = \sqrt{-|g(x)|} \left[ \frac{1}{16\pi G} [R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}(x) - 8\pi G T^{\alpha\beta}] \right] = 0 \quad (35)$$

where  $T^{\alpha\beta}$  is the energy-momentum tensor of matter field, which is a symmetrical (2,0)-tensor.

$$\begin{aligned}
T^{\alpha\beta} &\equiv \frac{2}{\sqrt{-|g(x)|}} \frac{\delta A_M[g, u]}{\delta g_{\alpha\beta}(x)} = g^{\alpha\beta}(x) \mathcal{L} + 2 \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}(x)} \Big|_{u(x), \nabla u(x)} \\
&\quad + 2 \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\xi^\theta(x)} \frac{\partial \nabla_\lambda u_\xi^\theta(x)}{\partial g_{\alpha\beta}(x)} - 2 \Gamma_{\nu\lambda}^\nu(x) \frac{\partial \mathcal{L}}{\partial \nabla_\mu u_\xi^\theta(x)} \frac{\partial \nabla_\mu u_\xi^\theta(x)}{\partial \partial_\lambda g_{\alpha\beta}(x)} \\
&\quad - 2 \partial_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\mu u_\xi^\theta(x)} \frac{\partial \nabla_\mu u_\xi^\theta(x)}{\partial \partial_\lambda g_{\alpha\beta}(x)} - 2 \frac{\partial \mathcal{L}}{\partial \nabla_\mu u_\xi^\theta(x)} \partial_\lambda \frac{\partial \nabla_\mu u_\xi^\theta(x)}{\partial \partial_\lambda g_{\alpha\beta}(x)} \\
&= T^{\alpha\beta}(u(x), \partial u(x), g(x), \partial g(x)) \quad (36)
\end{aligned}$$

All the pre-GR dynamics discuss how matter behaves in spacetime with a given metric, that is, Newton's absolute space-time or Minkowski space. All of them can not explain why there are inertial reference systems and non-inertial reference systems, and why the inertial mass equals the gravitational mass. G.R. is unique. It discusses how the matter motion determines the bending of the spacetime and how the matter behaves in a curved spacetime. Hence the dynamical variables are both the matter field  $u_\xi^\theta(x)$  and the metric field of spacetime  $g_{\alpha\beta}(x)$ . It means that they are both to be determined simultaneously by solving the equation of motion. It does not necessarily imply that the metric field of spacetime  $g_{\alpha\beta}(x)$  is matter (in its philosophic context, in the most general meaning of the word) like, say, the electromagnetic field, the spinor field, etc. It does not necessarily imply that the metric field of spacetime  $g_{\alpha\beta}(x)$  carries energy-momentum.

## 4.2 Noether's theorem for classical field in GR

The Noether's conservation law, or the continuity equation (26) , now reads

$$\frac{\partial}{\partial x^\kappa} \left\{ \sqrt{-|g(x)|} J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)] \right\} = 0$$

or

$$\int_{x(\partial\Omega)} ds_\lambda(x) \{ \sqrt{-|g(x)|} J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)] \} = 0 \quad (37)$$

where

$$\begin{aligned} J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)] &=: J_x^\kappa \\ &= \{ \mathcal{L} \delta_\rho^\kappa \delta x^\rho + \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} \bar{\delta} u_\xi^\theta(x) + \frac{1}{2} \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) \right. \\ &\quad \left. + \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \right] \end{aligned}$$

$$\begin{aligned}
& - \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) ] \bar{\delta} g_{\alpha\beta}(x) \} + \\
& \frac{1}{16\pi G} \{ R \delta_\rho^\kappa \delta x^\rho + [ \frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \\
& - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} ] \bar{\delta} g_{\alpha\beta}(x) + \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu \bar{\delta} g_{\alpha\beta}(x) \} \quad (38)
\end{aligned}$$

It is worth noting that the form of function  $J^\kappa$  is independent of coordinate systems, and the arguments of function  $J^\kappa$  are  $u(x)$ ,  $\partial u(x)$ ,  $g(x)$ ,  $\partial g(x)$ ,  $\partial^2 g(x)$ ; and  $\delta x$ ,  $\bar{\delta} u(x)$ ,  $\bar{\delta} g(x)$ ,  $\partial \bar{\delta} g(x)$ .

### 4.3 Conservation laws in GR

#### 4.3.1 Conservation law due to “coordinate shift” invariance

Action (33) remains unchanged under the following “coordinate shifts”.

$$\delta x^\rho = \epsilon^\rho, \quad \delta u_\xi^\theta(x) = 0, \quad \delta g_{\alpha\beta}(x) = 0. \quad (39)$$

In this case, eqn.(26) reads

$$\partial_\lambda [\sqrt{-|g(x)|} \tau_\rho^\lambda(x)] = 0, \quad (40)$$

where

$$\begin{aligned}
\tau_\rho^\kappa(x) &= \tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \\
&= \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} \partial_\rho u_\xi^\theta(x) - \mathcal{L} \delta_\rho^\kappa + \frac{1}{2} [ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) \\
&+ \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) \\
&- \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\
&+ \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) ] \partial_\rho g_{\alpha\beta}(x) + \frac{1}{16\pi G} \\
&[ ( \frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} ) \partial_\rho g_{\alpha\beta}(x) \\
&+ \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu \partial_\rho g_{\alpha\beta}(x) - R \delta_\rho^\kappa ] \quad (41)
\end{aligned}$$

is usually called canonical energy-momentum tensor. It is worth noting that the arguments of function  $\tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$  are all **field quantities and their derivatives**. Notice that "coordinate shift" eqn.(39) is not an invariant concept under general coordinate transformation. This explains why  $\tau_\sigma^\lambda(x)$  is not a tensor under general coordinate transformation. We will get back to this problem later.

#### 4.3.2 Conservation law due to "4-dimensional rotation" invariance

The action (33) remains unchanged under infinitesimal "4-dimensional rotations" (Lorentz transformations), which form a 6-parameter family of infinitesimal symmetry transformations

$$\begin{aligned} x^\mu &\longmapsto \tilde{x}^\mu = L_\nu^\mu x^\nu, \quad L_\nu^\mu = \delta_\nu^\mu + \Lambda_\nu^\mu, \quad |\Lambda_\nu^\mu| \ll 1, \quad \eta_{\mu\lambda}\Lambda_\nu^\lambda \equiv \Lambda_{\mu\nu}, \quad \Lambda_{\mu\nu} = -\Lambda_{\nu\mu}, \\ \delta x^\lambda &= \Lambda_\mu^\lambda x^\mu = \frac{1}{2}(\eta^{\lambda\rho}x^\sigma - \eta^{\lambda\sigma}x^\rho)\Lambda_{\rho\sigma}, \end{aligned} \quad (42)$$

$$\begin{aligned} \delta u_\xi^\theta(x) &= \Lambda_\varphi^\theta u_\xi^\varphi(x) - \Lambda_\xi^\eta u_\eta^\theta(x) \\ &= \Lambda_{\rho\sigma} \frac{1}{2} [\eta^{\theta\rho} u_\xi^\sigma(x) - \eta^{\theta\sigma} u_\xi^\rho(x) - \delta_\xi^\sigma \eta^{\eta\rho} u_\eta^\theta(x) + \delta_\xi^\rho \eta^{\eta\sigma} u_\eta^\theta(x)], \end{aligned} \quad (43)$$

$$\begin{aligned} \delta g_{\alpha\beta}(x) &= \Lambda_{\rho\sigma} \frac{1}{2} [-\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) \\ &\quad + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) - \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \end{aligned} \quad (44)$$

In this case eqn.(26) reads

$$\frac{\partial}{\partial x^\kappa} \{ \sqrt{-|g(x)|} M^{\kappa\rho\sigma} \} = 0 \quad (45)$$

where

$$\begin{aligned} &2M^{\kappa\rho\sigma}(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \\ &= \mathcal{L} \delta_\lambda^\kappa (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) + \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} \{ [\eta^{\theta\rho} u_\xi^\sigma(x) - \eta^{\theta\sigma} u_\xi^\rho(x) \\ &\quad - \delta_\xi^\sigma \eta^{\eta\rho} u_\eta^\theta(x) + \delta_\xi^\rho \eta^{\eta\sigma} u_\eta^\theta(x)] - (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \partial_\lambda u_\xi^\theta(x) \} \\ &\quad + \frac{1}{2} \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) \right] \end{aligned}$$



$$\begin{aligned}
& - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\
& - \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) ] \times \\
& \{ [-\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) - \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \\
& - (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x) \} + \frac{1}{16\pi G} \{ R \delta_\lambda^\kappa (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \\
& + (\frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)}) \\
& ([-\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) - \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \\
& - (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x)) + \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu ([-\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) \\
& + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) - \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \\
& - (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x)) \} \tag{46}
\end{aligned}$$

#### 4.3.3 Conservation law due to “4-dimensional pure deformation” invariance

The action (33) remains unchanged under infinitesimal “4-dimensional pure deformations”, which form a 6-parameter family of infinitesimal symmetry transformations

$$\begin{aligned}
x^\mu & \longmapsto \tilde{x}^\mu = L_\nu^\mu x^\nu, \quad L_\nu^\mu = \delta_\nu^\mu + \Lambda_\nu^\mu, \quad |\Lambda_\nu^\mu| \ll 1, \\
\eta_{\mu\lambda} \Lambda_\nu^\lambda & \equiv \Lambda_{\mu\nu}, \quad \Lambda_{\mu\nu} = \Lambda_{\nu\mu}, \tag{47}
\end{aligned}$$

$$\delta x^\lambda = \Lambda_\mu^\lambda x^\mu = \frac{1}{2} (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \Lambda_{\rho\sigma}, \tag{48}$$

$$\begin{aligned}
\delta u_\xi^\theta(x) & = \Lambda_\varphi^\theta u_\xi^\varphi(x) - \Lambda_\xi^\eta u_\eta^\theta(x) \\
& = \frac{1}{2} [\eta^{\theta\rho} u_\xi^\sigma(x) + \eta^{\theta\sigma} u_\xi^\rho(x) - \delta_\xi^\sigma \eta^{\eta\rho} u_\eta^\theta(x) - \delta_\xi^\rho \eta^{\eta\sigma} u_\eta^\theta(x)] \Lambda_{\rho\sigma}, \tag{49}
\end{aligned}$$

$$\begin{aligned}
\delta g_{\alpha\beta}(x) & = -\frac{1}{2} [\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) \\
& + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) + \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \Lambda_{\rho\sigma} \tag{50}
\end{aligned}$$

In this case eqn.(26) reads

$$\frac{\partial}{\partial x^\kappa} \{ \sqrt{-|g(x)|} N^{\kappa\rho\sigma} \} = 0 \quad (51)$$

where

$$\begin{aligned} & 2N^{\kappa\rho\sigma}(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \\ = & \mathcal{L} \delta_\lambda^\kappa (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) + \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} \{ [\eta^{\theta\rho} u_\xi^\sigma(x) + \eta^{\theta\sigma} u_\xi^\rho(x) \\ & - \delta_\xi^\sigma \eta^{\eta\rho} u_\eta^\theta(x) - \delta_\xi^\rho \eta^{\eta\sigma} u_\eta^\theta(x)] - (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \partial_\lambda u_\xi^\theta(x) \} \\ & + \frac{1}{2} \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) \right. \\ & - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\ & \left. - \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) \right] \times \\ & \{ [\delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) + \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x)] \\ & + (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x) \} + \frac{1}{16\pi G} \{ R \delta_\lambda^\kappa (\eta^{\lambda\rho} x^\sigma - \eta^{\lambda\sigma} x^\rho) \\ & - \left( \frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \right) \\ & [ \delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) + \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) \\ & + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x) + (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x) ] \\ & - \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu [ \delta_\alpha^\sigma \eta^{\mu\rho} g_{\mu\beta}(x) + \delta_\alpha^\rho \eta^{\mu\sigma} g_{\mu\beta}(x) + \delta_\beta^\sigma \eta^{\nu\rho} g_{\alpha\nu}(x) \\ & + \delta_\beta^\rho \eta^{\nu\sigma} g_{\alpha\nu}(x) + (\eta^{\lambda\rho} x^\sigma + \eta^{\lambda\sigma} x^\rho) \partial_\lambda g_{\alpha\beta}(x) ] \} \end{aligned} \quad (52)$$

#### 4.3.4 Conservation law due to “scaling” invariance

The action (33) remains unchanged under infinitesimal scaling transformations, which form a 1-parameter family of infinitesimal symmetry transformations

$$x^\lambda \longmapsto \tilde{x}^\lambda = e^\epsilon x^\lambda, \quad |\epsilon| \ll 1, \quad \delta x^\lambda = \epsilon x^\lambda, \quad \forall \lambda = 0, 1, 2, 3 \quad (53)$$

$$\delta u_\xi^\theta(x) = 0, \quad \delta g_{\alpha\beta}(x) = -2\epsilon g_{\alpha\beta}(x) \quad (54)$$

In this case eqn.(26) reads

$$\frac{\partial}{\partial x^\kappa} \{ \sqrt{-|g(x)|} S^\kappa \} = 0 \quad (55)$$

$$\begin{aligned} & S^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) \\ &= \mathcal{L} \delta_\rho^\kappa x^\rho - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} x^\rho \frac{\partial}{\partial x^\rho} u_\xi^\theta(x) - \frac{1}{2} \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\beta(x) \right. \\ &+ \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\xi^\theta(x)} g^{\theta\alpha}(x) u_\xi^\kappa(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\xi^\theta(x)} g^{\theta\kappa}(x) u_\xi^\beta(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\kappa u_\beta^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) \\ &- \frac{\partial \mathcal{L}}{\partial \nabla_\beta u_\kappa^\theta(x)} g^{\xi\alpha}(x) u_\xi^\theta(x) + \left. \frac{\partial \mathcal{L}}{\partial \nabla_\alpha u_\beta^\theta(x)} g^{\xi\kappa}(x) u_\xi^\theta(x) \right] [2g_{\alpha\beta}(x) + x^\rho \partial_\rho g_{\alpha\beta}(x)] \\ &+ \frac{1}{16\pi G} \left\{ R \delta_\rho^\kappa x^\rho - \left[ \frac{\partial R}{\partial \partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \right] \right. \\ &\left. [2g_{\alpha\beta}(x) + x^\rho \partial_\rho g_{\alpha\beta}(x)] - \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu [2g_{\alpha\beta}(x) + x^\rho \partial_\rho g_{\alpha\beta}(x)] \right\} \quad (56) \end{aligned}$$

#### 4.3.5 Conservation law due to “skew-scaling” invariance

The action (33) remains unchanged under infinitesimal “skew-scaling” transformations, which form a 1-parameter family of infinitesimal symmetry transformations

$$\delta x^0 = -\epsilon^1 x^0, \quad \delta x^1 = \epsilon^1 x^1, \quad \delta x^2 = 0, \quad \delta x^3 = 0 \quad (57)$$

$$\left[ \frac{\partial \tilde{x}}{\partial x} \right] = \begin{bmatrix} 1 - \epsilon^1 & 0 & 0 & 0 \\ 0 & 1 + \epsilon^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (58)$$

$$\begin{aligned} \delta u_0^0(x) &= 0, \quad \delta u_1^0(x) = -2\epsilon^1 u_1^0(x), \quad \delta u_2^0(x) = -\epsilon^1 u_2^0(x), \quad \delta u_3^0(x) = -\epsilon^1 u_3^0(x) \\ \delta u_0^1(x) &= 2\epsilon^1 u_0^1(x), \quad \delta u_1^1(x) = 0, \quad \delta u_2^1(x) = \epsilon^1 u_2^1(x), \quad \delta u_3^1(x) = \epsilon^1 u_3^1(x) \\ \delta u_0^j(x) &= \epsilon^1 u_0^j(x), \quad \delta u_1^j(x) = -\epsilon^1 u_1^j(x), \quad \delta u_k^j(x) = 0, \quad \forall j, k = 2, 3 \\ \delta g_{00}(x) &= 2\epsilon^1 g_{00}(x), \quad \delta g_{01}(x) = 0, \quad \delta g_{02}(x) = \epsilon^1 g_{02}(x), \quad \delta g_{03}(x) = \epsilon^1 g_{03}(x) \end{aligned}$$

$$\begin{aligned}\delta g_{10}(x) &= 0, \quad \delta g_{11}(x) = -2\epsilon^1 g_{11}(x), \quad \delta g_{12}(x) = -\epsilon^1 g_{12}(x), \quad \delta g_{13}(x) = -\epsilon^1 g_{13}(x) \\ \delta g_{j0}(x) &= \epsilon^1 g_{j0}(x), \quad \delta g_{j1}(x) = -\epsilon^1 g_{j1}(x), \quad \delta g_{jk}(x) = 0, \quad \forall j, k = 2, 3\end{aligned}\quad (59)$$

Substitute eqns.(57) and (59) into eqn.(26), we can get a conserved current

$$\frac{\partial}{\partial x^\kappa} \{ \sqrt{-|g(x)|} J_1^\kappa \} = 0 \quad (60)$$

Here we skip the expression of  $J_1^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$ . Similarly we can get conserved currents  $J_2^\kappa$  and  $J_3^\kappa$ .

#### 4.4 Equivalence principle and non-localizability

Because the symmetry group of classical field in GR is an infinite dimensional Lie group, we can get infinitely many conservation laws by using Noether's theorem. It is interesting to note that most of the conserved currents are not tensors. For instance,  $\tau_\rho^\kappa$  is not a (1,1) tensor,  $M^{\kappa\rho\sigma}$  is not a (3,0) tensor,  $N^{\kappa\rho\sigma}$  is not a (3,0) tensor, and  $S^\kappa$  is not a (1,0) tensor, etc. One of the consequences of the pseudotensor property is the non-localizability. Relativitists attribute the non-localizability of the gravitational energy-momentum to the equivalence principle. The argument goes as follows. In a freely falling spaceship, the astronaut feels no gravitational field acting on him; while people on the earth see that the gravity is acting on him to keep him in the orbit around the earth. Hence the existence of the gravitational field at a spacetime point depends on the reference coordinate system adopted; non-localizability of gravitational energy-momentum is the direct consequence of the equivalence principle; and it is inherent in the theory of GR. This sounds very reasonable. But, let us examine the following example, Landau-Lifshitz's gravitational energy-momentum pseudotensor [7]

$$\begin{aligned}t^{\mu\nu}(x) &= \frac{1}{16\pi G} \{ [2\Gamma_{\lambda\alpha}^\beta(x)\Gamma_{\beta\rho}^\rho(x) - \Gamma_{\lambda\rho}^\beta(x)\Gamma_{\alpha\beta}^\rho(x) - \Gamma_{\beta\lambda}^\beta(x)\Gamma_{\alpha\rho}^\rho(x)] \\ &\quad [g^{\mu\lambda}(x)g^{\nu\alpha}(x) - g^{\mu\nu}(x)g^{\lambda\alpha}(x)] + g^{\mu\lambda}(x)g^{\alpha\beta}(x)[\Gamma_{\lambda\rho}^\nu(x)\Gamma_{\alpha\beta}^\rho(x) + \\ &\quad \Gamma_{\alpha\beta}^\nu(x)\Gamma_{\lambda\rho}^\rho(x) - \Gamma_{\beta\rho}^\nu(x)\Gamma_{\lambda\alpha}^\rho(x) - \Gamma_{\lambda\alpha}^\nu(x)\Gamma_{\beta\rho}^\rho(x)] \\ &\quad + g^{\nu\lambda}(x)g^{\alpha\beta}(x)[\Gamma_{\lambda\rho}^\mu(x)\Gamma_{\alpha\beta}^\rho(x) + \Gamma_{\alpha\beta}^\mu(x)\Gamma_{\lambda\rho}^\rho(x) - \Gamma_{\beta\rho}^\mu(x)\Gamma_{\lambda\alpha}^\rho(x) \\ &\quad - \Gamma_{\lambda\alpha}^\mu(x)\Gamma_{\beta\rho}^\rho(x)] + g^{\lambda\alpha}(x)g^{\beta\rho}(x)[\Gamma_{\lambda\beta}^\mu(x)\Gamma_{\alpha\rho}^\nu(x) - \Gamma_{\lambda\alpha}^\mu(x)\Gamma_{\beta\rho}^\nu(x)] \}.\end{aligned}$$

When the energy-momentum tensor of matter vanishes  $t^{\alpha\beta}(x) \equiv 0$ , the Einstein field equation has the solution  $g_{\alpha\beta}(x) \equiv \eta_{\alpha\beta}$ . The spacetime is the Minkowski space, and the coordinate system  $(x^0, x^1, x^2, x^3)$  is a coordinate system of inertia. In this coordinate system,

$$t^{\mu\nu}(x) \equiv 0, \forall 0 \leq \mu, \nu \leq 3.$$

Let us switch to coordinate system  $(y^0, y^1, y^2, y^3) =: (t, r, \theta, \varphi)$ , such that

$$\begin{aligned} t &= x^0, r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \\ \theta &= \cos^{-1} \frac{x^3}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}, \\ \varphi &= \tan^{-1} \frac{x^2}{x^1}. \end{aligned}$$

In this spherical polar coordinates,

$$t^{00}(y) = \frac{-(3 + \cot^2 \theta)}{8\pi G r^2} < 0.$$

Note that the coordinate system  $(x^0, x^1, x^2, x^3)$  and the coordinate system  $(y^0, y^1, y^2, y^3) =: (t, r, \theta, \varphi)$ , unlike the above mentioned spaceship and earth, are not in relative motion; they share a common reference system. Yet for any spacetime point  $p \in M$ , we have  $t^{\mu\nu}(x) = 0$ ; and  $t^{00}(y) < 0$ . This counter-example shows that the non-localizability of gravitational energy-momentum in GR, is far beyond what can be simply attributed to the equivalence principle.

One of the main tasks of the present work is to show that these pseudotensor problems are just results from misreading Noether's theorem.

## 5 Noether's theorem revisited

Let us start with the simplest example. The canonical energy-momentum  $\tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$  in eqn.(41) is not a tensor

$$\begin{aligned} \tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x)) &\neq \\ \frac{\partial x^\kappa}{\partial y^\lambda} \frac{\partial y^\sigma}{\partial x^\rho} \tau_\sigma^\lambda(u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y)) &. \end{aligned} \quad (61)$$

This can be understood as follows. The 4-parameter families of infinitesimal transformations of coordinates and field quantities eqn.(39) and

$$\delta y^\rho = \epsilon^\rho, \quad \delta u_\xi^\theta(y) = 0, \quad \delta g_{\alpha\beta}(y) = 0. \quad (39')$$

do not correspond to the same 4-parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$ , when

$$\exists \quad p \in \mathbb{M}, \text{ and } 0 \leq \alpha, \beta, \mu \leq 3, \text{ such that } \frac{\partial^2 y^\mu}{\partial x^\alpha \partial x^\beta} \Big|_p \neq 0, \quad (62)$$

Therefore, generally speaking, the conservation laws (continuity equations) corresponding to them are not equivalent to each other, and the conserved quantities corresponding to them are not the same geometrical physical object. Inequality (61) is a comparison between components of two different geometrical physical objects which have been mistaken one and the same.

In general, infinitesimal transformations of coordinates and field quantities expressed in different coordinate systems with the same form (like eqns.(39) and (39')) do not correspond to the same infinitesimal diffeomorphism of spacetime  $\mathbb{M}$ . Conversely, the same  $r$ -parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$  is described differently in different coordinate systems. For instance, if a 4-parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$  is described in coordinate system  $(x^0, x^1, x^2, x^3)$  by eqn.(39), then it would be described in coordinate system  $(y^0, y^1, y^2, y^3)$  not by eqn.(39'), but by

$$\begin{aligned} \delta y^\rho &= \frac{\partial y^\rho}{\partial x^\mu} \epsilon^\mu, \quad \delta u_\xi^\theta(y) = \epsilon^\mu \left[ \frac{\partial}{\partial y^\varphi} \left( \frac{\partial y^\theta}{\partial x^\mu} \right) u_\xi^\varphi(x) - \frac{\partial}{\partial y^\xi} \left( \frac{\partial y^\theta}{\partial x^\mu} \right) u_\eta^\theta(x) \right] \\ \delta g_{\alpha\beta}(y) &= -\epsilon^\mu \left[ \frac{\partial}{\partial y^\beta} \left( \frac{\partial y^\sigma}{\partial x^\mu} \right) g_{\alpha\sigma}(x) + \frac{\partial}{\partial y^\alpha} \left( \frac{\partial y^\rho}{\partial x^\mu} \right) g_{\rho\beta}(x) \right] \end{aligned} \quad (39'')$$

which generally is no longer “coordinate shift”.

It is generally accepted that in GR, the same physical geometrical object is expressed in different coordinate systems by functions of **field quantities, their derivatives, and coordinates**, with the same function form. (For instance,  $\tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$  of (41) and  $\tau_\rho^\kappa(u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y))$  are generally considered the components of the geometrical physical object, canonical energy-momentum.) But this assertion is not always true. For instance, corresponding to the same  $r$ -parameter family of infinitesi-

mal diffeomorphisms of spacetime  $\mathbb{M}$ , the Noether's conserved current (38)  $J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)]$ , and its counterpart in coordinate system  $(y^0, y^1, y^2, y^3)$ ,

$J^\kappa [u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y); \delta y, \bar{\delta} u(y), \bar{\delta} g(y), \partial \bar{\delta} g(y)]$ , are not function of field quantities, their derivatives, and coordinates, with the same function form. (Because  $J^\kappa$  manifestly contains  $\delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)$ , besides field quantities, their derivatives and coordinates, while  $\delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)$  and  $\delta y, \bar{\delta} u(y), \bar{\delta} g(y), \partial \bar{\delta} g(y)$  are different functions of field quantities, and their derivatives, and coordinates.) Yet they are the same physical geometrical object, as will be shown in the following.

To prove this, let us show that continuity equations written in different coordinate systems, but corresponding to the same  $r$ -parameter family of infinitesimal diffeomorphisms of spacetime  $\mathbb{M}$ , are equivalent to each other, even though the conserved currents written in different coordinate systems are different functions of matter field, metric field, their derivatives, and coordinates.

Let  $\{\Delta(\epsilon^1, \dots, \epsilon^r) =: \Delta : \mathbb{M} \rightarrow \mathbb{M} \mid |\epsilon^i| \ll 1, \forall 1 \leq i \leq r\}$  be an  $r$ -parameter family of infinitesimal diffeomorphisms of spacetime manifold  $\mathbb{M}$  onto itself, and  $\Delta(0, \dots, 0)$  the identity mapping of  $\mathbb{M}$ . For any chart  $(U, \varphi)$  of  $\mathbb{M}$ , denoting by  $(x^0, x^1, x^2, x^3)$  the corresponding coordinate system, we have

$$(x^0, x^1, x^2, x^3) \xrightarrow{\pi^\lambda \circ \varphi \circ \Delta \circ \varphi^{-1}} \tilde{x}^\lambda = x^\lambda + \delta x^\lambda(x; \epsilon^1, \dots, \epsilon^r) =: x^\lambda + \delta x^\lambda, \quad (63)$$

where  $\pi^\lambda : (x^0, x^1, x^2, x^3) \mapsto x^\lambda$  is the projection operator,

$$\begin{aligned} \delta u_\xi^\theta(x; \epsilon^1, \dots, \epsilon^r) &= : \delta u_\xi^\theta(x) = \frac{\partial \delta x^\theta}{\partial x^\varphi} u_\xi^\varphi(x) - \frac{\partial \delta x^\eta}{\partial x^\xi} u_\eta^\theta(x), \\ \delta g_{\alpha\beta}(x; \epsilon^1, \dots, \epsilon^r) &= : \delta g_{\alpha\beta}(x) = -\frac{\partial \delta x^\mu}{\partial x^\alpha} g_{\mu\beta}(x) - \frac{\partial \delta x^\nu}{\partial x^\beta} g_{\alpha\nu}(x) \end{aligned} \quad (64)$$

and the continuity equation (37) which is re-written for simplicity as

$$\frac{\partial}{\partial x^\kappa} [\sqrt{-|g(x)|} J_x^\kappa] = 0 \quad (65)$$

Switching to coordinate system  $(y^0, y^1, y^2, y^3)$ , which corresponds to chart  $(V, \psi)$ , we have

$$(y^0, y^1, y^2, y^3) \xrightarrow{\pi^\lambda \circ \psi \circ \Delta \circ \psi^{-1}} \tilde{y}^\lambda = y^\lambda + \delta y^\lambda(y, \epsilon^1, \dots, \epsilon^r) =: y^\lambda + \delta y, \quad (63')$$

$$\begin{aligned}
\delta u_\xi^\theta(y; \epsilon^1, \dots, \epsilon^r) &= : \delta u_\xi^\theta(y) = \frac{\partial \delta y^\theta}{\partial y^\varphi} u_\xi^\varphi(y) - \frac{\partial \delta y^\eta}{\partial y^\xi} u_\eta^\theta(y), \\
\delta g_{\alpha\beta}(y; \epsilon^1, \dots, \epsilon^r) &= : \delta g_{\alpha\beta}(y) = -\frac{\partial \delta y^\mu}{\partial y^\alpha} g_{\mu\beta}(y) - \frac{\partial \delta y^\nu}{\partial y^\beta} g_{\alpha\nu}(y) \quad (64')
\end{aligned}$$

$$\frac{\partial}{\partial y^\kappa} \left[ \sqrt{-|g(y)|} J_y^\kappa \right] = 0 \quad (65')$$

For the same  $r$ -parameter family of infinitesimal diffeomorphisms of space-time manifold  $\mathbb{M}$  onto itself  $\{\Delta(\epsilon^1, \dots, \epsilon^r) =: \Delta : \mathbb{M} \rightarrow \mathbb{M} \mid |\epsilon^i| \ll 1, \forall 1 \leq i \leq r\}$ , we have

$$\delta y^\lambda = \frac{\partial(\pi^\lambda \circ \psi \circ \varphi)}{\partial x^\kappa} \delta x^\kappa = \frac{\partial y^\lambda}{\partial x^\kappa} \delta x^\kappa \quad (66)$$

From this relation it is easy to see that  $\delta x^\lambda(x; \epsilon^1, \dots, \epsilon^r)$  and  $\delta y^\lambda(y; \epsilon^1, \dots, \epsilon^r)$ , generally speaking, are functions of different forms. However, we will show that the continuity eqns. (65) and (65') are equivalent to each other.  $J_x^\kappa$  and  $J_y^\kappa$  are respectively the components in coordinate systems  $(x^0, x^1, x^2, x^3)$  and  $(y^0, y^1, y^2, y^3)$  of the same vector field.

#### Theorem 4

$$\frac{\partial}{\partial x^\kappa} [\sqrt{-|g(x)|} J_x^\kappa] = 0 \iff \frac{\partial}{\partial y^\kappa} [\sqrt{-|g(y)|} J_y^\kappa] = 0 \quad (67)$$

**Proof.** The lhs of eqn.(65) can be written as

$$\begin{aligned}
\frac{\partial}{\partial x^\kappa} [\sqrt{-|g(x)|} J_x^\kappa] &= \frac{\partial y^\lambda}{\partial x^\kappa} \frac{\partial}{\partial y^\lambda} \left[ \sqrt{-|g(y)|} \left\| \frac{\partial y}{\partial x} \right\| J_x^\kappa \right] \\
&= \frac{\partial y^\lambda}{\partial x^\kappa} \frac{\partial}{\partial y^\lambda} \left[ \sqrt{-|g(y)|} J_x^\kappa \right] \left\| \frac{\partial y}{\partial x} \right\| + \frac{\partial}{\partial x^\kappa} \left\| \frac{\partial y}{\partial x} \right\| \left[ \sqrt{-|g(y)|} J_x^\kappa \right] \\
&= \frac{\partial}{\partial y^\lambda} \left[ \sqrt{-|g(y)|} \frac{\partial y^\lambda}{\partial x^\kappa} J_x^\kappa \right] \left\| \frac{\partial y}{\partial x} \right\| - \left( \frac{\partial}{\partial y^\lambda} \frac{\partial y^\lambda}{\partial x^\kappa} \right) \left[ \sqrt{-|g(y)|} J_x^\kappa \right] \left\| \frac{\partial y}{\partial x} \right\| \\
&\quad + \frac{\partial^2 y^\lambda}{\partial x^\kappa \partial x^\alpha} \frac{\partial x^\alpha}{\partial y^\lambda} \left\| \frac{\partial y}{\partial x} \right\| \left[ \sqrt{-|g(y)|} J_x^\kappa \right] \\
&= \frac{\partial}{\partial y^\lambda} \left[ \sqrt{-|g(y)|} \frac{\partial y^\lambda}{\partial x^\kappa} J_x^\kappa \right] \left\| \frac{\partial y}{\partial x} \right\| \quad (68)
\end{aligned}$$

here an identity on the Jacobian (For its proof see appendix C)

$$\frac{\partial}{\partial x^\kappa} \left\| \frac{\partial y}{\partial x} \right\| = \frac{\partial}{\partial y^\lambda} \left( \frac{\partial y^\lambda}{\partial x^\kappa} \right) \left\| \frac{\partial y}{\partial x} \right\| \quad (69)$$



has been used. Hence the proof is reduced to proving

$$\frac{\partial y^\lambda}{\partial x^\kappa} J_x^\kappa = J_y^\lambda \quad (70)$$

In fact, it's easy to show that  $\delta x^\kappa$  is a vector,  $\bar{\delta} u_\xi^\theta(x) = \tilde{u}_\xi^\theta(x) - u_\xi^\theta(x) = \delta u_\xi^\theta(x) - \delta x^\rho \partial_\rho u_\xi^\theta(x)$  is a (1,1)-tensor, and  $\bar{\delta} g_{\alpha\beta}(x) = \tilde{g}_{\alpha\beta}(x) - g_{\alpha\beta}(x) = \delta g_{\alpha\beta}(x) - \delta x^\rho \partial_\rho g_{\alpha\beta}(x)$  is a (0,2)-tensor. Hence the terms in the first brace of  $J_x^\kappa$  eqn.(38) are vectors, and the first term in the second brace is a vector too. The rest of the terms in the second brace are not vectors individually. However, their sum is a vector. This can be proven straightforwardly, though tediously. (See Appendix B) ■

**Conclusion 5** *Now we have proven:*

(i) *It is not the Noether's conservation laws written in different coordinate systems corresponding to the infinitesimal coordinate transformations with the same form that are equivalent to one another; rather, it is the Noether's conservation laws corresponding to the same family of infinitesimal diffeomorphisms of spacetime onto itself that are equivalent to one another.*

(ii) *All the Noether's conservative currents are vector fields (eqn.(70)), which should be the density, and current density of some scalar. Therefore,*

(iii) *All the Noether's conserved quantities are scalars.*

It is worth noting that the form of function  $J^\kappa$  eqn.(38) as a function of  $u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x), \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)$  is independent of coordinate systems. However, for a given family of infinitesimal diffeomorphisms of spacetime, the function forms of  $\delta x(x, \epsilon^1, \dots, \epsilon^r), \bar{\delta} u(x, \epsilon^1, \dots, \epsilon^r), \bar{\delta} g(x, \epsilon^1, \dots, \epsilon^r), \partial \bar{\delta} g(x, \epsilon^1, \dots, \epsilon^r)$  and  $\delta y(y, \epsilon^1, \dots, \epsilon^r), \bar{\delta} u(y, \epsilon^1, \dots, \epsilon^r), \bar{\delta} g(y, \epsilon^1, \dots, \epsilon^r), \partial \bar{\delta} g(y, \epsilon^1, \dots, \epsilon^r)$  are generally different, hence the function forms of  $J_x^\kappa$  and  $J_y^\kappa$  as functions of field quantities, their derivatives, and coordinates, are different. This does not ruin the covariance of our presentation which is assured by the coordinate independence of eqn.(38).

We have proven these for the case of GR.. However it is easy to show they are true for the cases of SR and pre-relativity dynamics.

According to Einstein, “[w]hat we call physics comprises that group of natural sciences which base their concepts on measurements; and whose concepts and propositions lend themselves to mathematical formulation.” The aim of all natural sciences is to search for the **objective** laws of nature. In order to measure physical quantities or to describe physical processes in

terms of mathematics, one needs first to choose a reference coordinate system which depends on the observer's **subjective** will. Therefore, the ways of performing measurements and the formulations of physical laws, expressed in all reference coordinate systems should take the same form. This idea has guided Einstein from pre-relativity physics in Newtonian absolute space-time, to SR in Minkowski spacetime, and finally to GR in curved space-time. In particular, to get the proper motion equation one has to use an invariant Lagrangian  $\mathcal{L}(g(x), \partial g(x), \partial^2 g(x), u(x), \partial u(x))$ , such that the action  $A[g, u] = \int_{x(\Omega)} d^4x \sqrt{-|g(x)|} \mathcal{L}(g(x), \partial g(x), \partial^2 g(x), u(x), \partial u(x))$  is an invariant under general transformation of coordinates. The least action principle would lead to a motion equation independent of coordinate systems. The general covariance of a physical theory means the operations of measurement, and the mathematical expressions of physical laws should be independent of reference coordinate systems. It does not mean that functions of coordinates, field quantities, and their derivatives, in different coordinate systems but with the same function form, must represent the same physical geometrical object, as is generally accepted. This is not always true as has been shown above. So, Noether's conserved current (38)

$$J^\kappa [u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x); \delta x, \bar{\delta} u(x), \bar{\delta} g(x), \partial \bar{\delta} g(x)] =: J_x^\kappa \text{ and}$$

$J^\kappa [u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y); \delta y, \bar{\delta} u(y), \bar{\delta} g(y), \partial \bar{\delta} g(y)] =: J_y^\kappa$  are components of the same vector field on spacetime; while the canonical energy-momentum (41)  $\tau_\rho^\kappa(x) = \tau_\rho^\kappa(u(x), \partial u(x), g(x), \partial g(x), \partial^2 g(x))$  and  $\tau_\rho^\kappa(y) = \tau_\rho^\kappa(u(y), \partial u(y), g(y), \partial g(y), \partial^2 g(y))$  are not components of the same tensor field on spacetime. The long standing pseudotensor problems like (61), are only a result of misreading Noether's theorem; and the non-locality difficulty of gravitational energy-momentum does not really exist at all, had we read Noether's theorem properly. They are results of mistaking different physical geometrical objects as one and the same.

It is important to distinguish a general law of nature and a concrete physical object or process. The former is independent of coordinates, while the latter might look different to different observers (from different coordinate systems). Yet we still can observe and describe a concrete instance in a way independent of coordinate systems.

Now we are in a position to address the problems of conservation of energy-momentum and the gravitational energy-momentum in GR. These will be done in a later paper.

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## A Appendix

The Lagrangian density of classical fields  $L$ , is a function of the coordinates, field quantities, and their derivatives of up to the  $N$ -th order. However, because not all the variables are independent, such as  $\partial_\mu \partial_\nu u_\eta^\xi(x) = \partial_\nu \partial_\mu u_\eta^\xi(x)$ ,  $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ , etc., there are infinitely many different function forms for  $L$ . This causes indefiniteness of derivatives, such as  $\frac{\partial}{\partial \partial_\mu \partial_\nu u_\eta^\xi(x)} L$ ,  $\frac{\partial}{\partial g_{\alpha\beta}(x)} L$ . If we drop all the redundant variables, then the Einstein summation convention can no longer be used, and the expressions will become awfully complicated, especially for a large  $N$ . In order to keep the formulae neat, physicists usually treat it in a different way. Here we will illustrate their method by using the lagrangian density for vacuum Einstein's equation,  $R$  (Ricci's scalar curvature).

$R$  is a function of 16  $g_{\alpha\beta}(x)$ 's, 64  $\partial_\mu g_{\alpha\beta}(x)$ 's, and 256  $\partial_\mu \partial_\nu g_{\alpha\beta}(x)$ 's. Because  $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ , and  $\partial_\mu \partial_\nu g_{\alpha\beta}(x) = \partial_\nu \partial_\mu g_{\alpha\beta}(x)$ , there are only 150 independent variables among them. We will choose 10  $g_{\alpha\beta}(x)$ 's, 40  $\partial_\mu g_{\alpha\beta}(x)$ 's, and 100  $\partial_\mu \partial_\nu g_{\alpha\beta}(x)$ 's ( $\alpha \leq \beta, \mu \leq \nu$ ), for the independent variables. As a

function of 336 variables (As a function defined on a 336-dimensional domain),  $R$  can take infinite different forms, say,  $\varphi, \psi, \dots$ . When restricted to the 150-dimensional “sub-domain”  $D$ , all of them are the same function of 150 variables.

$$R|_D = \varphi|_D = \psi|_D = \dots \quad (\text{A1})$$

Substituting the 150 independent variables for all the variables in  $\varphi, \psi, \dots$ , we get a unique function

$$\underline{R}(g_{\alpha\beta}(x), \partial_\mu g_{\alpha\beta}(x), \partial_\mu \partial_\nu g_{\alpha\beta}(x)), (\alpha \leq \beta, \mu \leq \nu) \quad (\text{A2})$$

Substituting  $\frac{1}{2}(g_{\alpha\beta}(x) + g_{\beta\alpha}(x))$ ,  $\frac{1}{2}(\partial_\mu g_{\alpha\beta}(x) + \partial_\mu g_{\beta\alpha}(x))$ , and  $\frac{1}{4}(\partial_\mu \partial_\nu g_{\alpha\beta}(x) + \partial_\mu \partial_\nu g_{\beta\alpha}(x) + \partial_\nu \partial_\mu g_{\alpha\beta}(x) + \partial_\nu \partial_\mu g_{\beta\alpha}(x))$  for  $g_{\alpha\beta}(x)$ ,  $\partial_\mu g_{\alpha\beta}(x)$ , and  $\partial_\mu \partial_\nu g_{\alpha\beta}(x)$  in  $\underline{R}$ , respectively, we get a unique function of all 336 variables, denoted by  $R(g(x), \partial g(x), \partial^2 g(x))$ . This “standard”  $R(g(x), \partial g(x), \partial^2 g(x))$  has the following property.

$$\begin{aligned} \frac{\partial R}{\partial g_{\alpha\beta}(x)} &= \frac{\partial R}{\partial g_{\beta\alpha}(x)} = \frac{1}{2} \frac{\partial \underline{R}}{\partial g_{\alpha\beta}(x)}, \alpha < \beta \\ \frac{\partial R}{\partial g_{\alpha\alpha}(x)} &= \frac{\partial \underline{R}}{\partial g_{\alpha\alpha}(x)} \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \frac{\partial R}{\partial \partial_\mu g_{\alpha\beta}(x)} &= \frac{\partial R}{\partial \partial_\mu g_{\beta\alpha}(x)} = \frac{1}{2} \frac{\partial \underline{R}}{\partial \partial_\mu g_{\alpha\beta}(x)}, \alpha < \beta \\ \frac{\partial R}{\partial \partial_\mu g_{\alpha\alpha}(x)} &= \frac{\partial \underline{R}}{\partial \partial_\mu g_{\alpha\alpha}(x)} \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} &= \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} = \frac{\partial R}{\partial \partial_\nu \partial_\mu g_{\alpha\beta}(x)} = \\ \frac{\partial R}{\partial \partial_\nu \partial_\mu g_{\beta\alpha}(x)} &= \frac{1}{4} \frac{\partial \underline{R}}{\partial \partial_\mu g_{\alpha\beta}(x)}, \alpha < \beta, \mu < \nu \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \frac{\partial R}{\partial \partial_\mu \partial_\nu g_{\alpha\alpha}(x)} &= \frac{\partial R}{\partial \partial_\nu \partial_\mu g_{\alpha\alpha}(x)} = \frac{1}{2} \frac{\partial \underline{R}}{\partial \partial_\mu \partial_\nu g_{\alpha\alpha}(x)}, \mu < \nu \\ \frac{\partial R}{\partial \partial_\mu \partial_\mu g_{\alpha\beta}(x)} &= \frac{\partial R}{\partial \partial_\mu \partial_\mu g_{\alpha\beta}(x)} = \frac{1}{2} \frac{\partial \underline{R}}{\partial \partial_\mu \partial_\mu g_{\alpha\beta}(x)}, \alpha < \beta \\ \frac{\partial R}{\partial \partial_\mu \partial_\mu g_{\alpha\alpha}(x)} &= \frac{\partial \underline{R}}{\partial \partial_\mu \partial_\mu g_{\alpha\alpha}(x)} \end{aligned} \quad (\text{A6})$$

When calculating the derivatives of  $R$ , we pretend that all its 336 variables are independent. Thus the indefiniteness problem no longer exists.

From (A1), we have

$$\delta\varphi|_D = \delta\psi|_D$$

While

$$\begin{aligned}
\delta\varphi|_D &= \left[ \frac{\partial\varphi}{\partial g_{\alpha\beta}(x)} \delta g_{\alpha\beta}(x) + \frac{\partial\varphi}{\partial \partial_\mu g_{\alpha\beta}(x)} \delta \partial_\mu g_{\alpha\beta}(x) + \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} \delta \partial_\mu \partial_\nu g_{\alpha\beta}(x) \right] |_D \\
&= \left[ \sum_\alpha \frac{\partial\varphi}{\partial g_{\alpha\alpha}(x)} \delta g_{\alpha\alpha}(x) + \sum_{\alpha<\beta} \left( \frac{\partial\varphi}{\partial g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial g_{\beta\alpha}(x)} \right) \delta g_{\alpha\beta}(x) \right. \\
&\quad + \sum_\alpha \frac{\partial\varphi}{\partial \partial_\mu g_{\alpha\alpha}(x)} \delta \partial_\mu g_{\alpha\alpha}(x) + \sum_{\alpha<\beta} \left( \frac{\partial\varphi}{\partial \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\mu g_{\beta\alpha}(x)} \right) \delta \partial_\mu g_{\alpha\beta}(x) \\
&\quad + \sum_{\alpha,\mu} \frac{\partial\varphi}{\partial \partial_\mu g_{\beta\alpha}(x)} \delta \partial_\mu g_{\beta\alpha}(x) + \sum_{\alpha,\mu} \frac{\partial\varphi}{\partial \partial_\mu \partial_\mu g_{\alpha\alpha}(x)} \delta \partial_\mu \partial_\mu g_{\alpha\alpha}(x) \\
&\quad + \sum_{\alpha<\beta,\mu} \left( \frac{\partial\varphi}{\partial \partial_\mu \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\mu \partial_\mu g_{\beta\alpha}(x)} \right) \delta \partial_\mu \partial_\mu g_{\alpha\beta}(x) \\
&\quad + \sum_{\alpha,\mu<\nu} \left( \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\alpha\alpha}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\alpha\alpha}(x)} \right) \delta \partial_\mu \partial_\nu g_{\alpha\alpha}(x) \\
&\quad + \sum_{\alpha<\beta,\mu<\nu} \left( \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\alpha\beta}(x)} + \right. \\
&\quad \left. \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\beta\alpha}(x)} \right) \delta \partial_\mu \partial_\nu g_{\alpha\beta}(x) \Big] |_D \tag{A7}
\end{aligned}$$

Because all the variations on the RHS of (A7) are independent, we get

$$\begin{aligned}
\left( \frac{\partial\varphi}{\partial g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial g_{\beta\alpha}(x)} \right) |_D &= \left( \frac{\partial\psi}{\partial g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial g_{\beta\alpha}(x)} \right) |_D \\
\left( \frac{\partial\varphi}{\partial \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\mu g_{\beta\alpha}(x)} \right) |_D &= \left( \frac{\partial\psi}{\partial \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial \partial_\mu g_{\beta\alpha}(x)} \right) |_D \\
\left( \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\varphi}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} + \frac{\partial\varphi}{\partial \partial_\nu \partial_\mu g_{\beta\alpha}(x)} \right) |_D &= \left( \frac{\partial\psi}{\partial \partial_\mu \partial_\nu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial \partial_\nu \partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial \partial_\mu \partial_\nu g_{\beta\alpha}(x)} + \frac{\partial\psi}{\partial \partial_\nu \partial_\mu g_{\beta\alpha}(x)} \right) |_D
\end{aligned}$$

$$= (\frac{\partial\psi}{\partial\partial_\mu\partial_\nu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial\partial_\nu\partial_\mu g_{\alpha\beta}(x)} + \frac{\partial\psi}{\partial\partial_\mu\partial_\nu g_{\beta\alpha}(x)} + \frac{\partial\psi}{\partial\partial_\nu\partial_\mu g_{\beta\alpha}(x)})|_D \quad (\text{A8})$$

This tells us, say,

$$\begin{aligned} (\frac{\partial R}{\partial g_{\alpha\beta}(x)}\delta g_{\alpha\beta}(x))|_D &= (\frac{\partial\varphi}{\partial g_{\alpha\beta}(x)}\delta g_{\alpha\beta}(x))|_D = (\frac{\partial\psi}{\partial g_{\alpha\beta}(x)}\delta g_{\alpha\beta}(x))|_D \\ (\frac{\partial R}{\partial\partial_\mu g_{\alpha\beta}(x)}\delta\partial_\mu g_{\alpha\beta}(x))|_D &= (\frac{\partial\varphi}{\partial\partial_\mu g_{\alpha\beta}(x)}\delta\partial_\mu g_{\alpha\beta}(x))|_D \\ &= (\frac{\partial\psi}{\partial\partial_\mu g_{\alpha\beta}(x)}\delta\partial_\mu g_{\alpha\beta}(x))|_D \\ (\frac{\partial R}{\partial\partial_\mu\partial_\nu g_{\alpha\beta}(x)}\delta\partial_\mu\partial_\nu g_{\alpha\beta}(x))|_D &= (\frac{\partial\varphi}{\partial\partial_\mu\partial_\nu g_{\alpha\beta}(x)}\delta\partial_\mu\partial_\nu g_{\alpha\beta}(x)) \\ &= (\frac{\partial\psi}{\partial\partial_\mu\partial_\nu g_{\alpha\beta}(x)}\delta\partial_\mu\partial_\nu g_{\alpha\beta}(x))|_D \quad (\text{A9}) \end{aligned}$$

and even more, such as

$$\begin{aligned} &(\frac{\partial R}{\partial\partial_\kappa g_{\alpha\beta}(x)})(\delta g_{\alpha\beta}(x) - \delta x^\rho\partial_\rho g_{\alpha\beta}(x))|_D \\ &= (\frac{\partial\varphi}{\partial\partial_\kappa g_{\alpha\beta}(x)})(\delta g_{\alpha\beta}(x) - \delta x^\rho\partial_\rho g_{\alpha\beta}(x))|_D \end{aligned}$$

where  $R(g(x), \partial g(x), \partial^2 g(x))$  is the “standard” expression for  $R$ , and  $\varphi(g(x), \partial g(x), \partial^2 g(x))$  is any expression from (A1).

## B Appendix

**Proposition 7** *Let*

$$\begin{aligned} I_x^\kappa =: & (\frac{\partial R}{\partial\partial_\kappa g_{\alpha\beta}(x)} - \partial_\mu \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}(x)} - \Gamma_{\nu\mu}^\nu(x) \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}(x)}) \times \\ & (\delta g_{\alpha\beta}(x) - \delta x^\rho\partial_\rho g_{\alpha\beta}(x)) + \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}(x)} \partial_\mu (\delta g_{\alpha\beta}(x) - \delta x^\rho\partial_\rho g_{\alpha\beta}(x)). \quad (\text{B1}) \end{aligned}$$

*Then*

$$I_y^\lambda = \frac{\partial y^\lambda}{\partial x^\kappa} I_x^\kappa. \quad (\text{B2})$$

**Proof.**

$$R = g^{\alpha\beta}g^{\rho\sigma}(\partial_\alpha\partial_\rho g_{\beta\sigma} - \partial_\alpha\partial_\beta g_{\rho\sigma}) + g^{\alpha\beta}g^{\rho\sigma}g^{\xi\eta}(\partial_\alpha g_{\beta\rho}\partial_\sigma g_{\xi\eta} + \frac{3}{4}\partial_\alpha g_{\rho\xi}\partial_\beta g_{\sigma\eta} - \frac{1}{4}\partial_\xi g_{\alpha\beta}\partial_\eta g_{\rho\sigma} - \frac{1}{2}\partial_\rho g_{\alpha\xi}\partial_\eta g_{\beta\sigma} - \partial_\alpha g_{\beta\rho}\partial_\xi g_{\sigma\eta}) \quad (B3)$$

$$\frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}} = \frac{1}{2}(g^{\alpha\kappa}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\kappa}) - g^{\alpha\beta}g^{\kappa\mu} \quad (B4)$$

Note that  $\frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}}$  is a  $(4, 0)$ -tensor, symmetrical for  $(\kappa, \mu)$ , and for  $(\alpha, \beta)$ .

$$\begin{aligned} \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}} &= \partial_\mu g_{\xi\eta}[g^{\alpha\kappa}g^{\beta\mu}g^{\xi\eta} + g^{\alpha\beta}g^{\kappa\xi}g^{\mu\eta} + \frac{3}{2}g^{\alpha\xi}g^{\beta\eta}g^{\kappa\mu} \\ &\quad - \frac{1}{2}g^{\alpha\beta}g^{\kappa\mu}g^{\xi\eta} - g^{\alpha\xi}g^{\beta\mu}g^{\kappa\eta} - g^{\alpha\kappa}g^{\beta\xi}g^{\mu\eta} - g^{\alpha\xi}g^{\beta\kappa}g^{\mu\eta}] \\ &=: \partial_\mu g_{\xi\eta}B^{\kappa\alpha\beta\mu\xi\eta} \end{aligned} \quad (B5)$$

$$\begin{aligned} \partial_\mu \frac{\partial R}{\partial\partial_\kappa\partial_\mu g_{\alpha\beta}} &= \frac{1}{2}(\partial_\mu g^{\alpha\kappa}g^{\beta\mu} + \partial_\mu g^{\alpha\mu}g^{\beta\kappa}) - \partial_\mu g^{\alpha\beta}g^{\kappa\mu} \\ &\quad + \frac{1}{2}(g^{\alpha\kappa}\partial_\mu g^{\beta\mu} + g^{\alpha\mu}\partial_\mu g^{\beta\kappa}) - g^{\alpha\beta}\partial_\mu g^{\kappa\mu} \\ &= \partial_\mu g_{\xi\eta}[-\frac{1}{2}g^{\alpha\xi}g^{\eta\kappa}g^{\beta\mu} - \frac{1}{2}g^{\alpha\xi}g^{\eta\mu}g^{\beta\kappa} + g^{\alpha\xi}g^{\eta\beta}g^{\kappa\mu}] \\ &\quad + \partial_\mu g_{\xi\eta}[-\frac{1}{2}g^{\alpha\kappa}g^{\beta\xi}g^{\eta\mu} - \frac{1}{2}g^{\alpha\mu}g^{\beta\xi}g^{\kappa\mu} + g^{\alpha\beta}g^{\kappa\xi}g^{\eta\mu}] \\ &=: \partial_\mu g_{\xi\eta}A^{\kappa\alpha\beta\mu\xi\eta} \end{aligned} \quad (B6)$$

Note that  $A^{\kappa\alpha\beta\mu\xi\eta}$  and  $B^{\kappa\alpha\beta\mu\xi\eta}$  are  $(6, 0)$ -tensors. Let

$$\begin{aligned} C^{\kappa\alpha\beta\mu\xi\eta}(x) &=: B^{\kappa\alpha\beta\mu\xi\eta}(x) - A^{\kappa\alpha\beta\mu\xi\eta}(x) \\ &= -\frac{1}{2}g^{\alpha\xi}(x)g^{\beta\mu}(x)g^{\kappa\eta}(x) + \frac{1}{2}g^{\alpha\mu}(x)g^{\beta\xi}(x)g^{\kappa\eta}(x) \\ &\quad - \frac{1}{2}g^{\alpha\xi}(x)g^{\beta\kappa}(x)g^{\mu\eta}(x) - \frac{1}{2}g^{\alpha\kappa}(x)g^{\beta\xi}(x)g^{\mu\eta}(x) \\ &\quad + \frac{1}{2}g^{\alpha\xi}(x)g^{\beta\eta}(x)g^{\kappa\mu}(x) + g^{\alpha\kappa}(x)g^{\beta\mu}(x)g^{\xi\eta}(x) - \frac{1}{2}g^{\alpha\beta}(x)g^{\kappa\mu}(x)g^{\xi\eta}(x) \end{aligned} \quad (B7)$$

$$\Gamma_{\alpha\nu}^\nu(x) = \frac{\partial y^\beta}{\partial x^\alpha} \Gamma_{\beta\gamma}^\gamma(y) + \frac{\partial}{\partial y^\sigma} \left( \frac{\partial y^\sigma}{\partial x^\alpha} \right) \quad (\text{B8})$$

Then

$$\begin{aligned} I_x^\kappa &= [C^{\kappa\alpha\beta\mu\xi\eta}(x) \partial_\mu g_{\xi\eta}(x) - \Gamma_{\mu\nu}^\nu(x) \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}}] \bar{\delta} g_{\alpha\beta}(x) \\ &\quad + \frac{\partial R}{\partial \partial_\kappa \partial_\mu g_{\alpha\beta}(x)} \partial_\mu \bar{\delta} g_{\alpha\beta}(x) \\ &= \frac{\partial x^\kappa}{\partial y^{\kappa'}} [C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial x^\xi}{\partial y^{\xi'}} \frac{\partial x^\eta}{\partial y^{\eta'}} \frac{\partial}{\partial y^{\mu'}} \left( \frac{\partial y^{\xi''}}{\partial x^\xi} \frac{\partial y^{\eta''}}{\partial x^\eta} g_{\xi''\eta''}(y) \right) \\ &\quad - \Gamma_{\nu'\mu'}^{\nu'}(y) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)}] \bar{\delta} g_{\alpha'\beta'}(y) \\ &\quad + \frac{\partial x^\kappa}{\partial y^{\kappa'}} \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)} \frac{\partial x^\alpha}{\partial y^{\alpha'}} \frac{\partial x^\beta}{\partial y^{\beta'}} \frac{\partial}{\partial y^{\mu'}} \left[ \frac{\partial x^\alpha}{\partial y^{\alpha''}} \frac{\partial x^\beta}{\partial y^{\beta''}} \bar{\delta} g_{\alpha''\beta''}(y) \right] \\ &= \frac{\partial x^\kappa}{\partial y^{\kappa'}} \{ [C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \partial_{\mu'} g_{\xi'\eta'}(y) - \Gamma_{\nu'\mu'}^{\nu'}(y) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)}] \times \\ &\quad \bar{\delta} g_{\alpha'\beta'}(y) + \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)} \partial_{\mu'} \bar{\delta} g_{\alpha'\beta'}(y) \} + \frac{\partial x^\kappa}{\partial y^{\kappa'}} rest \\ &= \frac{\partial x^\kappa}{\partial y^{\kappa'}} I_y^{\kappa'} + \frac{\partial x^\kappa}{\partial y^{\kappa'}} rest \end{aligned}$$

where

$$\begin{aligned} rest &=: [-C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial^2 x^\xi}{\partial y^{\mu'} \partial y^{\xi'}} \frac{\partial y^{\xi''}}{\partial x^\xi} g_{\xi''\eta''}(y) \\ &\quad - C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\eta''}}{\partial x^\eta} g_{\xi''\eta''}(y) \\ &\quad + \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'\beta'}(y)}] \bar{\delta} g_{\alpha'\beta'}(y) \end{aligned} \quad (\text{B9})$$

We are going to show that *rest* vanishes. Its first term is

$$\begin{aligned} &-C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial^2 x^\xi}{\partial y^{\mu'} \partial y^{\xi'}} \frac{\partial y^{\xi''}}{\partial x^\xi} g_{\xi''\eta''}(y) \bar{\delta} g_{\alpha'\beta'}(y) \\ &= [g^{\alpha'\xi'}(y) g^{\beta'\kappa'}(y) g^{\mu'\eta'}(y) - \frac{1}{2} g^{\alpha'\xi'}(y) g^{\beta'\eta'}(y) g^{\kappa'\mu'}(y) \end{aligned}$$



$$-g^{\alpha'\kappa'}(y)g^{\beta'\mu'}(y)g^{\xi'\eta'}(y) + \frac{1}{2}g^{\alpha'\beta'}(y)g^{\kappa'\mu'}(y)g^{\xi'\eta'}(y)] \times \\ \frac{\partial^2 x^\xi}{\partial y^{\mu'} \partial y^{\xi'}} \frac{\partial y^{\xi''}}{\partial x^\xi} g_{\xi''\eta''}(y) \bar{\delta} g_{\alpha'\beta'}(y)$$

$$= g^{\alpha'\xi'}(y)g^{\beta'\kappa'}(y) \frac{\partial}{\partial x^\xi} \left( \frac{\partial x^\xi}{\partial y^{\xi'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{A})$$

$$- \frac{1}{2} g^{\alpha'\xi'}(y)g^{\kappa'\mu'}(y) \frac{\partial^2 x^\xi}{\partial y^{\mu'} \partial y^{\xi'}} \frac{\partial y^{\beta'}}{\partial x^\xi} \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{B})$$

$$-g^{\alpha'\kappa'}(y)g^{\beta'\mu'}(y) \frac{\partial}{\partial x^\xi} \left( \frac{\partial x^\xi}{\partial y^{\mu'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{A})$$

$$+ \frac{1}{2} g^{\alpha'\beta'}(y)g^{\kappa'\mu'}(y) \frac{\partial}{\partial x^\xi} \left( \frac{\partial x^\xi}{\partial y^{\mu'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad ((\text{C})) \quad (\text{B10})$$

The second term is

$$-C^{\kappa'\alpha'\beta'\mu'\xi'\eta'}(y) \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\eta''}}{\partial x^\eta} g_{\xi''\eta''}(y) \bar{\delta} g_{\alpha'\beta'}(y) \\ = [g^{\alpha'\xi'}(y)g^{\beta'\kappa'}(y)g^{\mu'\eta'}(y) - \frac{1}{2}g^{\alpha'\xi'}(y)g^{\beta'\eta'}(y)g^{\kappa'\mu'}(y)$$

$$-g^{\alpha'\kappa'}(y)g^{\beta'\mu'}(y)g^{\xi'\eta'}(y) + \frac{1}{2}g^{\alpha'\beta'}(y)g^{\kappa'\mu'}(y)g^{\xi'\eta'}(y)] \times \\ \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\eta''}}{\partial x^\eta} g_{\xi''\eta''}(y) \bar{\delta} g_{\alpha'\beta'}(y)$$

$$= g^{\beta'\kappa'}(y)g^{\mu'\eta'}(y) \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\alpha'}}{\partial x^\eta} \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{D})$$

$$- \frac{1}{2} g^{\beta'\eta'}(y)g^{\kappa'\mu'}(y) \frac{\partial^2 x^\eta}{\partial y^{\mu'} \partial y^{\eta'}} \frac{\partial y^{\alpha'}}{\partial x^\eta} \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{B})$$

$$-g^{\alpha'\kappa'}(y)g^{\beta'\mu'}(y) \frac{\partial}{\partial x^\eta} \left( \frac{\partial x^\eta}{\partial y^{\mu'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{A})$$

$$+ \frac{1}{2} g^{\alpha'\beta'}(y)g^{\kappa'\mu'}(y) \frac{\partial}{\partial x^\eta} \left( \frac{\partial x^\eta}{\partial y^{\mu'}} \right) \bar{\delta} g_{\alpha'\beta'}(y) \quad (\text{C}) \quad (\text{B11})$$

The third term is

$$\begin{aligned}
& \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha' \beta'}(y)} \bar{\delta} g_{\alpha' \beta'}(y) \\
&= \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) (g^{\alpha' \kappa'} g^{\beta' \mu'} - g^{\alpha' \beta'} g^{\kappa' \mu'}) \bar{\delta} g_{\alpha' \beta'}(y) \\
&= \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) g^{\alpha' \kappa'} g^{\beta' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{A}) \\
&\quad - \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\mu}{\partial y^{\mu'}} \right) g^{\alpha' \beta'} g^{\kappa' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{C}) \quad (\text{B12})
\end{aligned}$$

The forth term is

$$\begin{aligned}
& - \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha'' \beta'}(y)} \frac{\partial^2 x^\alpha}{\partial y^{\mu'} \partial y^{\alpha''}} \frac{\partial y^{\alpha'}}{\partial x^\alpha} \bar{\delta} g_{\alpha' \beta'}(y) \\
&= - \frac{\partial^2 x^\alpha}{\partial y^{\mu'} \partial y^{\alpha''}} \frac{\partial y^{\alpha'}}{\partial x^\alpha} (g^{\alpha'' \kappa'} g^{\beta' \mu'} - g^{\alpha'' \beta'} g^{\kappa' \mu'}) \bar{\delta} g_{\alpha' \beta'}(y) \\
&= - \frac{\partial^2 x^\alpha}{\partial y^{\mu'} \partial y^{\alpha''}} \frac{\partial y^{\alpha'}}{\partial x^\alpha} g^{\alpha'' \kappa'} g^{\beta' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{B}) \\
&\quad + \frac{\partial^2 x^\alpha}{\partial y^{\mu'} \partial y^{\alpha''}} \frac{\partial y^{\alpha'}}{\partial x^\alpha} g^{\alpha'' \beta'} g^{\kappa' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{B}) \quad (\text{B13})
\end{aligned}$$

The last term is

$$\begin{aligned}
& - \frac{\partial R}{\partial \partial_{\kappa'} \partial_{\mu'} g_{\alpha' \beta''}(y)} \frac{\partial^2 x^\beta}{\partial y^{\mu'} \partial y^{\beta''}} \frac{\partial y^{\beta'}}{\partial x^\beta} \bar{\delta} g_{\alpha' \beta'}(y) \\
&= - \frac{\partial^2 x^\beta}{\partial y^{\mu'} \partial y^{\beta''}} \frac{\partial y^{\beta'}}{\partial x^\beta} (g^{\alpha' \kappa'} g^{\beta'' \mu'} - g^{\alpha' \beta''} g^{\kappa' \mu'}) \bar{\delta} g_{\alpha' \beta'}(y) \\
&= - \frac{\partial^2 x^\beta}{\partial y^{\mu'} \partial y^{\beta''}} \frac{\partial y^{\beta'}}{\partial x^\beta} g^{\alpha' \kappa'} g^{\beta'' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{D}) \\
&\quad + \frac{\partial^2 x^\beta}{\partial y^{\mu'} \partial y^{\beta''}} \frac{\partial y^{\beta'}}{\partial x^\beta} g^{\alpha' \beta''} g^{\kappa' \mu'} \bar{\delta} g_{\alpha' \beta'}(y) \quad (\text{B}) \quad (\text{B14})
\end{aligned}$$

All the terms marked (A) cancel each other, all the terms marked (B) cancel each other, etc. Therefore we get

$$I_x^\kappa = \frac{\partial x^\kappa}{\partial y^{\kappa'}} I_y^{\kappa'} \quad (\text{B15})$$

That is (B2). ■

## C Appendix

**Proposition 8**

$$\frac{\partial}{\partial x^\kappa} \left| \frac{\partial y}{\partial x} \right| = \frac{\partial}{\partial y^\lambda} \left( \frac{\partial y^\lambda}{\partial x^\kappa} \right) \left| \frac{\partial y}{\partial x} \right| \quad (\text{C1})$$

**Proof.**

$$\begin{aligned} \frac{\partial}{\partial x^\kappa} \left| \frac{\partial y}{\partial x} \right| &= \frac{\partial}{\partial x^\kappa} \left( \frac{\partial y^\lambda}{\partial x^\alpha} \right) \frac{\partial}{\partial \left( \frac{\partial y^\lambda}{\partial x^\alpha} \right)} \left| \frac{\partial y}{\partial x} \right| \\ &= \frac{\partial^2 y^\lambda}{\partial x^\kappa \partial x^\alpha} \frac{\partial x^\alpha}{\partial y^\lambda} \left| \frac{\partial y}{\partial x} \right| \\ &= \frac{\partial}{\partial y^\lambda} \left( \frac{\partial y^\lambda}{\partial x^\kappa} \right) \left| \frac{\partial y}{\partial x} \right| \end{aligned} \quad (\text{C2})$$

■